Discrete Math and Theory 1: Combinatorics

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DMT I

Note sheets will be passed around. If you prefer to access them online, you can find one here.

1. The How many sequences question on the back of this review sheet is helpful for question 1 in the homework

Overview

Sequences

$$(1, 2, 3), (a, b, c)$$

A sequence is defined as an enumerated collection of objects in which repetitions are allowed and order matters.

Permutations

A reordering of a sequence:

Valid permutations of $(1, 2, 3)$:

$$(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)$$

Valid permutations of $(1, 2, 2)$:

$$(1, 2, 2), (2, 1, 2), (2, 2, 1)$$

Factorials

$$\prod_{n=1}^{n} i = (1)(2)(3)...(n-2)(n-1)(n) = n!$$

A short hand for multiplication. Simplify factorials through division, and $0! = 1$. Grow very rapidly.

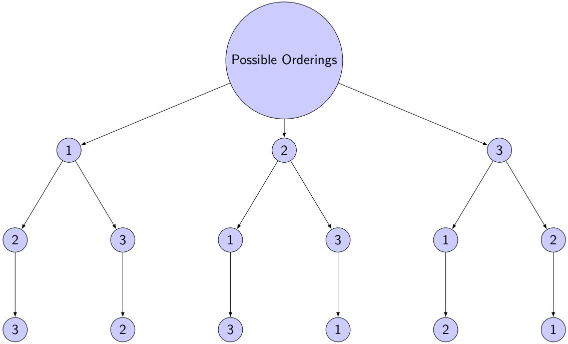
Number of ways to order n distinct elements

$$n!$$

n choices for the first element, $n-1$ choices for the second element, and so on until, 1 choice for the last element in our ordering.

The number of ways to order $(1, 2, 3)$

$$n = 3, \quad \rightarrow \quad 3!$$



Each route from the root to a leaf represents a possible ordering.

Number of ways to order n items with repeats

$$\frac{n!}{|a!||b!|...|y!||z!|}$$

For some set $S = \{(x_1, x_2, ..., x_n)|x \in \{a, b, c, ..., y, z\} \wedge |S| = n\}$
With $|a|, |b|, ..., |y|, |z|$ representing the frequency of that character.

Number of ways to order n items with repeats

mississippi

$$(mi_1 s_1 s_2 i_2 s_3 s_4 i_3 p_1 p_2 i_4) = (mi_2 s_1 s_2 i_1 s_3 s_4 i_3 p_1 p_2 i_4)$$

To reduce over counting we divide by the number of rearrangements, of each letter.

$$\frac{11!}{1!4!4!2!} = 69300$$

Number of ways to order $< n$ distinct items

$$^{n}P_k = \frac{n!}{(n-k)!} = (n)(n-1)(n-2)\cdots(n-k+1)$$

Read n Permute k. In the numerator, we calculate the number of ways to order n distinct items. In the denominator we undo the inclusion of $(n-k)$ items.

k elements out of n

How many different 5 card arrangements can you have in a deck of cards if you draw without replacement?

$$^{52}P_5 = \frac{(52)(51)(50)(49)(48)...(2)(1)}{(1)(2)...(47)} = (52)(51)(50)(49)(48)$$

This answer makes intuitive sense, because we have 52 options for the first card, then 51, then 50, 49, and finally 48.

Sets

A set is defined as an unordered collection of objects in which repetitions are not allowed and order does not matter.

$$\{1, 2, 3, 4\} = \{4, 3, 2, 1\} \qquad \{a, b, c\} = \{b, c, a\}$$

When answering questions regarding sets, we are concerned with the number of sets we can create. This is what a combination seeks to describs.

Keeping Sequences and Sets Straight

Sequences correspond to permutations and sets correspond to combinations. To keep this straight in my head, I think of a bike chain/lock.

I remind myself that combination locks should be called permutation locks. This is because order matters. For a 4 digit bike lock if 4321 is not your code and 1234 is, 4321 should not unlock the lock.

Notice that $\{1, 2, 3, 4\} = \{4, 3, 2, 1\}$ but $(4, 3, 2, 1) \neq (1, 2, 3, 4)$.

Combinations

$$_nC_k = \frac{n!}{(n-k)!k!} = \frac{(n)(n-1)...(n-k+1)}{k!} = \binom{n}{k}$$

Read n Choose k. We have k free choices starting from n possibilities, this is represented by $\frac{n!}{(n-k)!}$. Dividing by $k!$, guarantees that for each $(x_1, x_2, ..., x_k)$ sequence, we don't count a shuffling of this sequence.

Combination Problem 1

How many ways are there to choose a subset with cardinality 30 out of a set with cardinality 40?

$$n = 40, k = 30$$
$$_{40}C_{30} = \frac{40!}{30!10!} = 847660528$$

Summary

n^k	# total Sequences	Order Matters
$n!$	# orderings for n distinct elements	Order Matters
$\frac{n!}{a!b!\ldots z!}$	# Ways to rearrange a word	Order Matters
$\frac{n!}{(n-k)!}$	# Ways to order k out of n	Order Matters
$\frac{n!}{k!(n-k)!}$	# Ways to select a subset from a set	Does not Matter

Problem #1

A freshman class consists of 40 students, 30 of which are women. The class needs to select a committee of 7 to represent them in the student senate. How many committees are possible?

Problem #2

A freshman class consists of 40 students, 30 of which are women. The class needs to select a committee of 7 to represent them in the student senate. How many possible committees are there if there must be exactly 5 women on the committee?

Problem #3

A freshman class consists of 40 students, 30 of which are women. The class needs to select a committee of 7 to represent them in the student senate. How many possible committees are there if there must be at most 5 women on the committee?

Problem #4

A local United States telephone number has 7 digits and cannot start with 0, 1, or the three digits 555. How many such telephone numbers are possible?

Problem #5

How many sequences are there with exactly 8 characters taken from the 26 lower-case ASCII letters and 10 ASCII digits, with no characters repeated twice back-to-back (i.e., abadaeag is OK but abcddefg is not)

Problem #6

Exactly 8 characters taken from the 26 lower-case ASCII letters and 10 ASCII digits, with no repeated characters (i.e., neither abcdaefg nor abcddefg are OK)

Problem #7

How many unique shufflings of "alabama" are there?

Problem # 8

Imagine a state that requires default license plates to have 3 characters followed by 4 numbers: How many possible plates?

The Cardinality of the Powerset

The cardinality for the power set of a given set S, is: $\mathcal{P}(S) = 2^n$, for $n = |S|$. This is seen most clearly in the following proof by induction:

Base case: Let $n = 0$. The powerset of the emptyset is just a set containing the emptyset. $|\{\emptyset\}| = 1 = 2^0$ as desired.

Inductive Hypothesis:
Assume for some set S with $|S| = k$ and $k \geq 0$, that $\mathcal{P}(S) = |2^k|$

Inductive Step:
Given S we want to show that $S \cup \{x\} = S'$ for some element x, implies $|\mathcal{P}(S')| = 2^{n+1}$.

Inductive Step Continued:

The procedure for showing this is to write $\mathcal{P}(S)$ in rows. Then copy this column and add x to each element. Doing so doubles the number of copies, and since $2 \cdot 2^n = 2^{n+1}$, we can conclude that $|\mathcal{P}(S')| = 2^{n+1}$, hence by mathematical induction: $|\mathcal{P}(S)| = 2^n$.

Let $S = \{a, b, c\}$ $|S| = 3$ Assume by I.H. $|\mathcal{P}(S)| = 8$

$$
\begin{array}{rl}
\{a\} & \{a, x\} \\
\{b\} & \{b, x\} \\
\{c\} & \{c, x\} \\
\{a, b\} & \{a, b, x\} \\
\{a, c\} & \{a, c, x\} \\
\{b, c\} & \{b, c, x\} \\
\{a, b, c\} & \{a, b, c, x\} \\
\{\emptyset\} & \{x\}
\end{array}
$$

$S' = \{a, b, c, x\}$ $|S'| = 4$ and $|\mathcal{P}(S')| = 16$

Property 1 of Combinations

$$\binom{n}{k} = \binom{n}{n-k}$$

This hopefully becomes clear by inspection. When selecting k elements from a group of n we automatically make another group of $n-k$ elements (these are the elements left over).

$$S = \{A, B, C, D, E\}$$
$$\binom{5}{2} = \binom{5}{5-2}$$

$$\begin{array}{ccccccc} AB & \rightarrow & CDE & \quad & AC & \rightarrow & BDE \\ AD & \rightarrow & BCE & & AE & \rightarrow & BCD \\ BC & \rightarrow & ADE & & BD & \rightarrow & ACE \\ BE & \rightarrow & ACD & & CD & \rightarrow & ABE \\ CE & \rightarrow & ABD & & DE & \rightarrow & ABC \end{array}$$

Property 2 of Combinations (using the powerset)

$$\Sigma_{k=0}^{n} \binom{n}{k} = 2^n$$

This mathematical equation I hope makes intuitive sense, if we add up all of the number of subsets of size 1, all of the number of subsets of size 2, ..., to all of the subsets of size n, it should make intuitive sense that we get 2^n or the size of the powerset for a given set S!