MATH 3310

Contents

These are my notes for Professor Gazaki's Basic Real Analysis Fall 2024 Class. You can find them both on the course canvas and on my github here: https://github.com/HugoBarnes/LaTeX .

Chapter 1

Course Information

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1.1 Syllabus

Course Components. 2 Midterms + Final. Each midterm is 22.5%. The final is 35%. All in class. The HW is 15%. Quizzes are the remaining 5%. Quizzes are given during discussion roughly every second week. Quizzes are quick and fast. Makes sure you are on top of the class. Anywhere between 5-15 minutes. Discussion is mandatory as a result. Lowest quiz grade will be dropped. There is a course piazza created for this class. Use Piazza to post Questions. The MCLC might be offering one hour per week in Clemons 2. There will probably be a quiz next week.

Chapter 2

Sets of Numbers

Definition 2.0.1: Axioms of Addition

Let R be a number set such as $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

1. Closure under addition: If $x, y \in R$ then: $x + y \in R$
2. Commutativity under addition: $\forall x, y \in R$, we have $x + y = y + x$
3. Associativity under addition: $\forall x, y, z$, we have $x + (y + z) = (x + y) + z$.
4. Existence of additive identity: There exists $0 \in R s.t. \forall x \in R$, we have, $x + 0 = x$.
5. Existence of additive inverse: For all $x \in R \exists - x \in R s.t. x + (-x) = 0$.

Example 2.0.1 (Proof of Uniqueness)

The general sketch for a proof of uniqueness is to assume that what you are trying to prove is not unique and then there must be at least two different possible values. Then through algebraic manipulation, show that these two values must in fact be the same.

Proof: Let R be a set of numbers like $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$.

$$\text{For all } x \in R, \text{ let } a, b, \text{ be additive inverses to } x. \text{ Additive inverses are unique for: } a = a + 0$$
$$a = a + (x + b)$$
$$a = (a + x) + b$$
$$a = 0 + b$$
$$a = b$$

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Definition 2.0.2: Axioms of Multiplication

Let R be a number set such as $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

1. Closure under multiplication: If $x, y \in R$, then $x \cdot y \in R$.
2. Commutativity under multiplication: If $x, y \in R$, then $x \cdot y = y \cdot x$.
3. Associativity: If $x, y, z \in R$, then $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.
4. Existence of multiplicative identity: There exists $1 \in R$, such that $\forall x \in R, 1 \cdot x = x$.
5. Existence of multiplicative inverse: For all $x \in R$, there exists $\frac{1}{x} \in R$, such that, $x \cdot \frac{1}{x} = 1$.

Example 2.0.2 (Thought Experiment on Number Sets)

The simplest idea is to begin with the numbers that are the most simple. These are the counting numbers. They begin at **1**. $\mathbb{N} = \{1, 2, 3, ...\}$. We can add two numbers within this set and we can also multiply two numbers within this set. But this is where we are limited. Anything else, that isn't a derivative of multiplication and addition takes our numbers out of their set. The next thing to add is the negatives. This translates to adding the subtractions forming: $\mathbb{Z} = \{-2, -1, 0, 1, 2\}$. Again this creates problems as with $\mathbb{Z}$, we do not have access to division. Again we can create a new number set that accounts for this defined as : $\mathbb{Q} = \{\frac{m}{n}$ where $m, n \in \mathbb{Z}$ and $n \neq 0\}$. However this number set has gaps in it. For example the $\sqrt{2}$ is an irrational number,(see proof below). We define the $\mathbb{R}$ then as all of the numbers on the real number line that fill in the gaps that $\mathbb{Q}$ misses, and $\mathbb{Q}$.

Definition 2.0.3: Distributive Law

Let R be a set such as $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

1. $\forall x, y, z \in R$, we have $x \cdot (y + z) = x \cdot y + x \cdot z$.
2. $\forall x, y, z \in R$, we have $(x \cdot y) \cdot z = x \cdot z + y \cdot z$.

Definition 2.0.4: Positive Numbers

Let $P = \{a \in \mathbb{R}, \mathbb{Q}, \mathbb{Z} : a > 0\}$ Then $\forall a \in$ the above sets, the following will hold:

1. $a \in P$, a is a positive number
2. $-a \in P$, a is a negative number
3. $a = 0$, a is neither positive nor negative.

Two more axioms follow from the previous definition.

Definition 2.0.5: Axioms with Positives

1. if $a, b \in P$ then $a + b \in P$
2. if $a, b \in P$, then $a \cdot b > 0$.

Definition 2.0.6: Order

$\mathbb{Q}$ has a defined order. For any two elements within $\mathbb{Q}$, call them a, b one and only one of the following is true:

1. $a > b$
2. $a = b$
3. $a < b$

Theorem 2.0.1

$\sqrt{2}$ is an irrational number

Proof: We proceed by contradiction. Assume that $\sqrt{2}$ is a rational number. $\sqrt{2} = \frac{m}{n}$, where $m, n \in \mathbb{Z}$ and $n \neq 0$. We can also assume that m, n have no factors in common, that is that $\frac{m}{n}$ is a reduced fraction. By algebra we have:

$$2 = \frac{m^2}{n^2}$$

$$2n^2 = m^2$$

This implies that m^2 is even. By the following proof we show that this implies that m is even too:

Proof: We currently have $m^2 \longrightarrow m$ is even. It is enough to show that if m is odd we have m^2 is odd. This is the contrapositive statement. If m is odd then it is defined as $m = 2k + 1$ for some $k \in \mathbb{Z}$. $m^2 = 4k^2 + 4k + 1$ $m^2 = 2(2k^2 + 2k) + 1$. Because we are working with integers we can define $2k^2 + 2k$ to be an integer, call it l, then we can redefine $m^2 = 2l + 1$. Thus m is odd implies m^2 is odd.

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Because we know that m is even we can redefine it as $m = 2a$ where $a \in \mathbb{Z}$. The following is algebra.

$$2 = \frac{(2a)^2}{n^2}$$
$$\frac{4a^2}{n^2}$$
$$2n^2 = 4a^2$$
$$n^2 = 2a^2$$

This implies that n is even for the same reason that we found m to be even. Therein lies our contradiction. We assumed at the beginning that m, n had no factors in common. Thus $\sqrt{2}$ is an irrational number.

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2.1 A Summary of Today's Class

Started with the naturals. Then the Integers. Than the Rationals. We found that there are some issues with each of these sets. Potentially we will construct the real numbers. These are not the only sets of numbers that we can write down. Other sets of numbers: $S = \{1, 2, 4, 9\}$. We showed that we can write down sets by describing what the general element of the set looks like: $T = \{n \in \mathbb{N} : n \text{is even}\}$ or even $T = \{n \in \mathbb{N} s.t. n = 2k \text{ for some } k \in \mathbb{Z}\}$. Lastly, $B = \{x \in a : x^2 < 2\}$. We have operations with sets as well. The union and the intersection of the two sets.

Chapter 3

Thursday August 29th

3.1 Review from discussion

1. Equality between two Sets
2. Complement of a set

We will do everything you need for the HW.

3.1.1 Quick Review about function

Definition 3.1.1: What is a function

Given two sets A and B, a function f from A to B is a rule or mapping that associates each $x \in A$ and associates with it a unique element $y = f(x) \in B$. A = Domain, and B = Codomain.

1. All of the elements in the domain, A need to map somewhere to the second set B
2. It is okay to leave empty dots in B, we do not need to cover all of B. In other words the range does not have to equal B.
3. An example of something that is not a function would be one element of the first set going to two different points in B. You can not have an element of the first set map to two different elements.
4. The other thing that is not allowed is leaving elements in A undefined by $f(x)$.

3.1.2 Notation

$f : A \to B$
A is called the domain of the function. B is not necessarily the range of the function. B will be considered as a big target set, possibly larger than the range. Can always take B to be as large as possible.

Definition 3.1.2: Actual Range

Range of f is the set denoted
$f(A) = \{y \in B \text{ such that } y = f(x) \text{ for some } x \in A\}$

3.2 Some examlpes of function

Example 3.2.1 (Absolute value function)

$f : \mathbb{R} \to \mathbb{R}$
$f(x) = |x|$. The range is $\{x \in \mathbb{R}, x \geqslant 0\}$

Example 3.2.2 (Example 2)

$g(x) = \sqrt{1-x^2}$ Range is not everything.

Note:-

Functions don't always have to be a nice continuous graph

Example 3.2.3 (Example)

$f : \mathbb{N} \to \mathbb{R}, f(n) = n + 1$

Example 3.2.4 (Dirichlet's Funciton)

Define $f(x)$ to be one if x is a rational number and 0 if x is not a rational number.

3.3 Triangle Inequality

Definition 3.3.1: Absolute Value Function

The absolute value function: $f(x) = |x| = x$ if $x \geqslant 0, x = x$ otherwise if if $x < 0$, $-x \to x$.
The absolute value function has the following properties.

1. $|a \cdot b| = |a| \cdot |b|$ for all $a, b \in R$. Proceed this by doing cases. Positive negative and zero. If they have the same sign or opposite sign. Don't forget 0. Left as practice.
2. Triangle inequality (Super Important) $|a + b| \leqslant |a| + |b| \forall a, b \in \mathbb{R}$.
3. Analogs of the previous property. $|a - b| \leqslant |a| + |b| \forall a, b \in \mathbb{R}$
4. For $a, b, c \in \mathbb{R}$, we have $|a - b| \leqslant |a - c| + |c - b|$
5. Inverse Triangle inequality: $||a| - |b|| \leqslant |a - b|$

3.4 Proof for properties 3,4 of Triangle Inequality

Proof: Proof of property 3 assuming that property 2 is true.

Try to replace b with $-b$. Taking $|a+b| \leqslant |a|+|b|$. This means that we can plug in $-b$ directly because property 2 is true for all $a, b \in \mathbb{R}$. It must be the case that it remains true if we replace b with $-b$. $|a+(-b)| \leqslant |a|+|(-b)|$
$|a - b| \leqslant |a| + |b|$. Note that $|-b| = |b|$

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Proof: Proof of property 4.

Apply property 3. Plug in $a - c$ and $c - b$. Into $|a + b| \leqslant |a| + |b|$. Apply property 2 with $a - c$ in the place of a and with $c - b$ in the place of b.

$$|a + b| \leqslant |a| + |b|$$

Let $a - c = a$, and $b - c = b$. $|a - c + c - b| \leqslant |a - c| + |c - b|$

Using property 2 above.

$$|a - b| \leqslant |a - c| + |c - b|$$

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Theorem 3.4.1 Two numbers are equal

Let $a, b \in \mathbb{R}$, then $a = b$ if and only if the following is true, For every positive number $\epsilon > 0$ the absolute value of $|a - b| < \epsilon$.

Given 2 mathematical statements, what does it mean when ”if and only if”. Notation is $A \leftrightarrow B$. We also say that the statements A, B are equivalent. If A then B happens and equally if B then A.

In symbols this means $A \longrightarrow B$ and $B \longrightarrow A$. This means we need to do two separate proofs. The first step is to recognize what are the two directions.

Proof: Statement A: If we have $a = b$ then it must be the case that we have, for any $\epsilon > 0, |a - b| < \epsilon$
Statement B: If for every $\epsilon > 0$, we have $|a - b| < \epsilon$, then it must be the case that $a = b$.

Proof of Statement A. $A \longrightarrow B$. We know that $a = b$. We want to show that for every $\epsilon > 0$, the $|a - b| < \epsilon$. If $a = b$ and ϵ is assumed to be positive then we know that $|a - b| = 0$. This is true.
Proof of Statement B. $B \longrightarrow A$. We know that for every $\epsilon > 0$, $|a - b| < \epsilon$. We want to show that $a = b$.
We proceed by contradiction. We assume that $a \neq b$ and that $\epsilon > 0, |a - b| < \epsilon$ is true. Note that we only negate $a = b$.
If $b > a \longrightarrow b - a > 0$. If $a \neq b \longrightarrow a - b \neq 0$. This implies that $|a - b| > 0$. The assumption allows us to pick any $\epsilon > 0$ we like. In particular: for $\epsilon = |a - b| > 0$, we have $|a - b| < \epsilon$, is NOT TRUE. This is our contradiction.
Thus our initial assumption was false, and $a = b$.

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Proof:

Theorem 3.4.2 Let $a, b \in \mathbb{R}$

Then, $a = b \leftrightarrow$ for every $\epsilon > 0$, $|a - b| \leqslant \epsilon$.

Statement $A \longrightarrow B$ is the same. For Statement $B \longrightarrow A$. Our previous example no longer does the job because $|a - b| \leqslant \epsilon$ is still True. However any ϵ strictly smaller than $|a - b|$ will do the job.
Proof of $B \longrightarrow A$. Assume towards contradiction. That $a \neq b$, but for every $\epsilon > 0, |a - b| \leqslant \epsilon$.
$a - b \longrightarrow |a - b| > 0$. We pick ϵ to be $\frac{|a-b|}{2}$
Similarly we show that for an ϵ greater than 0, namely $\frac{|a-b|}{2}$, we have an epsilon less than $|a - b|$. This means that our orignial assumption was wrong, that $a = b$.

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3.5 Induction Proofs

Next Wednesday will be the first quiz. It will be a proof by induction. The proof on Tuesday will be very similar to the one we do on Wednesday on the Quiz.

Induction is a method to prove statements about the natural numbers. It is very restricted. It only applies to the $\mathbb{N}$. It is the domino proscess.

Example 3.5.1 (Induction Proof 1)

Theorem 3.5.1 For every $n \in \mathbb{N}$

$1 + 2 + \ldots + n = \frac{n(n+1)}{2}$

If we can prove that this is true for $n = 1$ and that it is true for $n + 1$. Then we have proved that this is true for all n.

Steps for for Induction.

1. Verify the base case, Verify that the statement is true.
2. (Induction Hypothesis): We assume that the statement is true for some $n \geqslant 1$. Take this for granted.
3. (Induction Step): We prove the statement for $n+1$. Using the induction hypothesis.

Proof: Proof by Induction.

Step one is to verify the base case for $n = 1$.

$$1 = \frac{1 \cdot (2)}{2} = 1$$

Step two is the induction hypothesis. Assume that $1 + 2 + \ldots + n = \frac{n \cdot (n+1)}{2}$ for some $n \geqslant 1$.

Step three is to prove the statement for $n + 1$.

We want to show: $1 + 2 + \ldots + (n+1) = \frac{(n+1)\cdot(n+2)}{2}$

$$1 + 2 + \ldots + n + n + 1 = \frac{n \cdot (n+1)}{2} + (n+1)$$

$$\frac{n(n+1)+(2n+2)}{2}$$

$$\frac{n^2+n+2n+2}{2}$$

$$\frac{(n+1)\cdot(n+2)}{2}$$

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Example 3.5.2 (The Bernoulli Inequality)

For every real number, $x \in \mathbb{R}$ with the property that $x \geqslant -1$, we have $(1+x)^n \geqslant 1 + (n \cdot x), \forall n \in \mathbb{N}$. We need to prove the second statement under the assumption that $x \geqslant -1$.

Proof: Prove the Bernoulli inequality with induction.

Let $x > -1$ and $n \in \mathbb{Z}^+$.

The base case for $n = 1$ holds as follows: $1 + x = 1 + (1 \cdot x) = 1 + x$.

Inductive Hypothesis: Assume that for some $k \geqslant 1$, the Bernoulli equation is true.

Inductive Step: We must show that $(1+x)^{k+1} \geqslant 1 + (k+1)x$.

Proof of inductive Step:

$$(1+x)^{k+1} = (1+x)(1+x)^k$$
$$\geqslant (1+x)(1+kx)$$
$$\geqslant kx^2 + kx + x + 1$$
$$\geqslant kx^2 + (k+1)x + 1$$
$$\geqslant 1 + (k+1)x,$$

since $kx^2 > 0$, this completes the induction step.

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Chapter 4

Tuesday September 3rd

4.1 Advertise the Directed Reading Program

Progam that gives you the chance to take a course with a graudate student or a postdoc. A baby step to doing math research. The deadline is September 6th.

4.2 Other announcements

Other office Hours are tomorrow from 3:30 - 5:00. and Thursday after class. The first quiz will be tomorrow. We will do an example today which will be very similar.

4.3 Finishing Up Induction

There was a gap in the Bernoulli Example:

Example 4.3.1 (Bernoulli Inequality)

For every real number $x \in \mathbb{R}$, with $x > -1$, we have $(1+x)^n \geqslant 1 + nx$, for all $n \in \mathbb{N}$. Why do we need the assumption that $x \geqslant -1$?
When we did the inductive step:
$(1+x)^{n+1} = (1+x)^n \cdot (1+x)$
This is only true if we have $x > -1$, if x is negative and we multiply by a negative inequality we flip the negative sign.

Example 4.3.2 (An Inductive Sequence)

Consider the following sequence:
x_n with $x_1 = 6$, and $x_{n+1} = \frac{2x_n - 6}{3}$. We get a recursive relation that relates x_n with x_{n+1} for $n \geqslant 1$.
Let's prove by induction that x_n is greater than or equal to x_{n+1} for all $n \in \mathbb{N}$.
Step 1 is to verify for $n = 1$: We want to show that $x_1 \geqslant x_2$. We can compute
$x_1 = 6, x_2 = 2$.
Step 2 is the induction hypothesis. We assume that $x_n \geqslant x_{n+1}$ for some $n \geqslant 1$.
Step 3 is the inductive step: We need to prove that $x_{n+1} \geqslant x_{n+2}$. Either we can start from $x_n \geqslant x_{n+1}$, and prove the plus one case. Or we can start at the end and simplify and work backward:

Proof: Start with $x_n \geqslant x_{n+1}$ and build $x_{n+1} \geqslant x_{n+2}$.

Step one is to multiply both sides by 2.

$$\begin{aligned} x_n &\geqslant x_{n+1} \\ 2x_n &\geqslant 2x_{n+1} \\ 2x_n - 6 &\geqslant 2x_{n+1} - 6 \end{aligned}$$

$\frac{2x_n-6}{3} \geqslant \frac{2x_{n+1}-6}{3}$

We are done here because the LHS equals x_{n+1} and the right hand side equals $xn+2$. Be careful about dividing by a negative number and flipping the inequality.

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4.4 Review From the First Lecture

We discussed that the set $\mathbb{Q}$, of rational numbers defined $\{\frac{m}{n} : m, n \in \mathbb{Z}\}$, satisfied 12 axioms. Axioms 1 through four were the addition axioms. Including Associativity and commutativity as well as identity. Five through 8 Were the multiplication axioms. With multiplicative identity and and associativity and commutativity and multiplicative inverse. We also discussed Distributivity, and the ordering axioms.
We discussed that the rational numbers have too many holes. We need to add one extra axiom to create the real numbers.

The set $\mathbb{R}$ of the real numbers satisfies the 12 axioms of the rational numbers and one more. The final axiom is called the axiom of completeness.

Definition 4.4.1: Axiom of Completeness

Every non-empty set $A \subseteq \mathbb{R}$ which is bounded above, has a least upper bound.

This creates two questions, what does it mean to be bounded above? and what is a least upper bound?

Definition 4.4.2: Bounded Above

A set $A \subseteq \mathbb{R}$ is called bounded above if there exists some $b \in \mathbb{R}$ such that $a \leqslant b$ for all $a \in A$. Any such number b is called an upper bound $\mathbb{U}B$ of A.

Definition 4.4.3: Bounded Below

A set $A \subseteq \mathbb{R}$ with $A \neq \emptyset$ is called bounded below if there exissts some $b \in \mathbb{R}$ such that $a \geqslant b$ for all $a \in A$. Any such number b is called a lower bound $\mathbb{L}B$ over A.

Example 4.4.1 (Bounded Above Example 1)

Take the set of $\mathbb{N}$ is only bounded below. The lower bound of the natural numbers is 0. Another example could be 1. Or $-\sqrt{2}$. $\mathbb{N}$ is not bounded above. Probably 1 is the best lower bound. It is the highest of all of the lower bounds.

Example 4.4.2 (Interval)

Take the interval, $I = (0, 1]$. This interal is bounded both above and below. A good upper bound could be 1. A good lower bound could be 0. As long as we find one upper bound or one lower bound we can create infinitely many by decreasing or increasing them.

Example 4.4.3 (Set Example)

$A = \{\frac{1}{n} : n \in \mathbb{N}$ both bounded above and below. 1 is an upper bound and 0 is a lower bound.

Note:-

If $A \subseteq \mathbb{R}$ is bounded above then A has infinitely many upper bounds. The same is true for bounded below

Definition 4.4.4: Least Upper Bound

Suppose $A \subseteq \mathbb{R}$, $A \neq \emptyset$ is bounded above. A real number $s \in \mathbb{R}$ is called a least upper bound (or supremum) of A if:

1. s is an upper bound of A in other words $s \geqslant a$ for all $a \in A$
2. s is the least upper bound of A, meaning if b is any upper bound of A, then it must be the case that $b \geqslant s$.

Definition 4.4.5: Greatest Lower Bound

Suppose $A \subseteq \mathbb{R}$, $A \neq \emptyset$, bounded below, a real number $t \in \mathbb{R}$ is called a greatest lower bound (or infimum) of A if:

1. t is a lower bound of A, meaning that for all $a \in A$ $t \leqslant a$.
2. t is the greatest lower bound for all upper bounds. Meaning given two upper bound b, t, it must be the case that $t \geqslant b$.

Note:-

In homework 2 using the above, we will show that every $B \subseteq \mathbb{R}$ $B \neq \emptyset$ bounded below has infimum.

Theorem 4.4.1 Upper Bound

Let $A \subseteq \mathbb{R}$, $A \neq \emptyset$, and A is bounded above. Then A has a unique least upper bound.The same thing applies in the reverse direction.

Proof: Let us assume that A has at least 2 least upper bounds, s_1, s_2. We take existence for granted from the axiom. We will show that the two must be the same. This is how we will always prove uniqueness.
We only need to apply the definition that s_1 and s_2 are least upper bounds. We will show that $s_1 \leqslant s_2$ and that $s_2 \leqslant s_1$. If both of these are simulataneously true, then they must be the same.
s_1 is an upper bound of A and s_2 is a least upper bound of A. Applying our definition of upperbound, we know that this fact implies that $s_1 \geqslant s_2$. Now we flip the roles. Similarly, $s_2$is an upper bound of A and s_1 is a least upper bound of A. Applying our definition of upperbound, property 2, we see that $s_2 \geqslant s_1$. This tells us that $s_1 \geqslant s_2 \wedge s_2 \geqslant s_2$. Both of these facts imply that $s_1 = s_2$. Thus s must be unique.

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Note:-
Notation

If $A \neq \emptyset$, $A \subseteq \mathbb{R}$ bounded above then $s = \sup A$, the least upper bound of A.
If $A \neq \emptyset$, $A \subseteq \mathbb{R}$ bounded below, then $t = \inf A =$ the greatest lower bound of A

Note:-
A supremum is never ifinity

We will never have $\sup A = \infty$

Note:-
See how $\sqrt{2}$ appears

We will take the set $A = \{x \in \mathbb{Q} : x^2 < 2\}$. This is a non empty bounded above subset of the real numbers. For example $1 \in A$. A is bounded above because 2 is an upper bound.
The axiom of completeness guarantees that this set has a least upper bound. Or that supremum may exists.
Guess that $\sup A = \sqrt{2}$.
The takeaway is that $\sqrt{2} \notin \mathbb{Q}$, which means that the supremum is outside of our set.

Chapter 1 is perhaps the hardest chapter in the course. If you can survive the first chapter. There is nothing to worry about. Chapter 1 is brutal, supremum, axiom of completeness.

Note:-

Difference between supremum and a maximum

Sometimes sets do not have a maximum number, you can place more decimal places to get a larger and larger number. This intuitively tells us that a maximum is good when it exists but it does not always exist.

Definition 4.4.6: Maximum Number

Let $A \neq \emptyset$, $A \subseteq \mathbb{R}$. A real number M is called a maximum of A if:

1. M is an upper bound of A, meaning $M \geqslant a$ for all $a \in A$
2. $M \in A$

Note:-

Similarly for minimum

Example 4.4.4

If $A \subseteq \mathbb{R}$, has a maximum $M \in A$, then $M = \sup A$.

Proof: Let M be the largest element in the set A, and let $\sup A$ be the supremum of A. By definition we know that for all elements in A, $M \geqslant a$ for all $a \in A$. This means that M is an uper bound of A. Additionally, if we take b to be a generic supremum of A we know that $b \geqslant M$ because M defines the largest element within A, this means that there can be no element less than M that is greater than all of the elements within A. This implies that $M \leqslant b$. Which provides definition two of a least upper bound. Thus M is a least upper bound.

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Note:-

Other approach

Contradiction is something that we only do when there is no other way to argue.

Note:-

Teacher Proof

Proof: s is the least upper bound of A, $M \in A \to M \leqslant s$. We can proceed in the other direction too. By definition of M being a maximum we know M has to be an upper bound. If M is an upper bound, and s is the least, this implies that $s \leqslant M$. We use that $M \in A$ and M is an upper bound.

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Example 4.4.5 (Fraction Example)

Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. There will be a quiz next week and you will have to show that A is bounded above and below and find supremum A and infimum of A. Some parts are easy and quick.

First we verify that A is not empty, bounded above and bounded below. We need to provide an element of the set, an upper bound, and a lower bound.
$A \neq \emptyset$: For example $\frac{1}{2} \in A$.
A is bounded below for example, -1 is a lower bound. $n \geqslant 1 \to \frac{1}{n} > 0 > -1$.
The same thing for bounded above: For example 1 is an upper bound of A, since $n \geqslant 1 \to \frac{1}{n} \leqslant 1$.
Now we need to check for a maximum and minimum. If it has a maximum we know the supremum and infimum.

The supremum of $A = 1$ because 1 is the maximum, 1 is an upper bound and an element of A which implies that 1 is a maximum and thus a supremum.
The guess is that the infimum should be zero. Zero is not the minimum because it is not in the set. We will justify this next time.

Chapter 5

Thursday September 5th

Important Lecture

5.1 Review From last time

Definition 5.1.1: Axiom of Completeness

Every non empty bounded above subset of the real numbers, has a least upper bound, (or supremum), and we showed last time that $S = \sup A$ is unique. S satisfies the following:

1. s is an upper bound of A, meaning that $s \geqslant a_1$ for all $a \in A$
2. If b is any upper bound of A, then $s \leqslant b$.

Note:-

How do we show $s = \sup A$ if $s \notin A$:

1. The first way to show that s satisfies 1,2 above
2. Show that s satisfies 1 above and if $b \in \mathbb{R}$ with $b < s$, then b is not an upper bound of the set A. This is equivalent to showing 2.

Class Answer: If $b \in \mathbb{R}$, with $b < s$ then there exists $a \in A$, with $a > b$.

Definition 5.1.2: Textbook Definition of Archimedian Property:

Theorem 5.1.1 Archimedian Property

1. Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying $n > x$.
2. Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $\frac{1}{n} < y$.

Textbook proof for the Archimedian Property:

Proof: Assume, for contradiction, that $\mathbb{N}$ *is* bounded above. By the axiom of completeness(AoC), $\mathbb{N}$ should then have a least upper bound, and we can set $\alpha = \sup \mathbb{N}$. If we consider $\alpha - 1$, then we no longer have an upper bound (Lemma 1.3.8), and therefore there exists an $n \in \mathbb{N}$ satisfying $\alpha - 1 < n$. But this is equivalent to $\alpha < n + 1$. Because $n + 1 \in \mathbb{N}$, we have a contradiction to the fact that α is supposed to be an upper bound for $\mathbb{N}$. (Notice that the contradiction here depends only on the axiom of completeness and the fact that $\mathbb{N}$ is closed under addition).
Part two follows from one by letting $x = \frac{1}{y}$.

😊

Lenma 5.1.1 Lemma 1.3.8 From the Textbook

Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then, $s = \sup A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $s - \epsilon < a$.

Proof: Here is a short rephrasing of the lemma: Given that s is an upper bound, s is the least upper bound if and only if any number smaller than s is not an upper bound.
($\rightarrow$) Forward direction, we assume that $s = \sup A$ and consider $s - \epsilon$, where $\epsilon > 0$ has been arbitrarily chosen. Becuase $s - \epsilon < s$, part two of the definition for a least upper bound implies that $s - \epsilon$ is not an upper bound for A. If this is the case, then there must be some element $a \in A$ for which $s - \epsilon < a$, (otherwise $s - \epsilon$ would be an upper bound). This proves the lemma in the forward direction.
($\leftarrow$) Conversely, assume s is an upper bound with the property that no matter how $\epsilon > 0$ is chosen, $s - \epsilon$ is no longer an upper bound for A. Notice that what this implies is that if b is any number less than s, then b is not an upper bound. (Just let $\epsilon = s - b$.) To prove that $s = \sup A$, we must verify part two of the definition for least upper bound. Because we have argued that any number smaller than s cannot be an upper bound, it follows that if b is some other uppder bound for A, then $s \leqslant b$.

😊

Note:-

ϵ Criterion for supremum:

$s = \sup A$ if and only if:

1. s is an upper bound of A
2. For all $\epsilon > 0$, the number $s - \epsilon$ is not an upper bound of A

Note:-

ϵ Criterion for $t = \inf A$

$t = \inf A$ if and only if:

1. t is a lower bound of A
2. For all $\epsilon > 0$, the number $t + \epsilon$ is not a lower bound of A, meaning that there is $a \in A$ such that $a < t + \epsilon$.

Example 5.1.1 (Exercise 1)
$A = \{\frac{1}{n} : n \in \mathbb{N}\}$. Prove that $\inf A = 0$.

Proof: We know that 0 is a lower bound of A. To show that it is the greatest lower bound of A, we need to show that for all $\epsilon > 0$, there is an element in our set, such that $a < 0 + \epsilon$. This is from the criterion of inf.
Because the elements of our set are defined by A we have:
Rewrite: We want to show that for all $\epsilon > 0$, there is an element of the set A or $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$.
This we have already proved. See above textbook definition for $A.P.$

😊

Next week there will be a quiz similar to the following question.

Example 5.1.2 (Exercise 2: QUIZ EXAMPLE)
Let $B = \{1 - \frac{2}{n+1} : n \in \mathbb{N}\}$
If I ask you to find everything, bounded above, below, supremum, infimum. Many will be easy, one will be hard.

1. $B \neq \emptyset$: $1 - \frac{2}{6} \in B$
2. B is bounded above: 1 is an upper bound of B because $\frac{2}{n+1} > 0 \to 1 - \frac{2}{n+1} < 1$.
3. B is bounded below. As an example 0 is a lower bound. In fact 0 is a minimum.
   $n \geqslant 1 \to n + 1 \geqslant 2$
   Because $n + 1 > 0$, we can divide both sides by $n + 1$, giving us: $1 \geqslant \frac{2}{n+1}$
   $\to 1 - \frac{2}{n+1} \geqslant 0$
   $\to 1 - \frac{2}{n+1} \geqslant 0 \to 0$ is a lower bound of B. But also for $n = 1$: $1 - \frac{2}{n+1} = 0$, so $0 \in B$. This shows the infimum of B is zero.
4. We know that 1 is an upper bound: $\sup B = 1$. Seen above.
   Want to show that 1 is the least upper bound, which means that we need to show for all $\epsilon > 0$, there is an element $b \in B$ such that $b > 1 - \epsilon$. This is the epsilon criteria for the supremum.

   Note:-
   Scratch Work

   Our proof will start always with let $\epsilon > 0$. We look for an element of the form of B, $\{1 - \frac{2}{n+1} : n \in \mathbb{N}\}$ to be bigger than $1 - \epsilon$. We are looking for an element in the set such that this statement is true.
   We simplify this as much as we can so that we can show the third definition of Archimedian Property.
   $1 - \frac{2}{n+1} > 1 - \epsilon$
   $\frac{-2}{n+1} > -\epsilon$
   $\frac{2}{n+1} < \epsilon$
   $\frac{1}{n+1} < \frac{\epsilon}{2}$
   Because $\frac{\epsilon}{2}$ is a positive number, we can define the A.P. for $\epsilon' = \frac{\epsilon}{2} > 0$.

   Proof: Let $\epsilon > 0$. Set $\epsilon' = \frac{\epsilon}{2}$. By the archimedian property applied for ϵ', we get that there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon'$
   This means that $\frac{1}{n} < \frac{\epsilon}{2}$. Since $\frac{1}{n+1}$ is less than $\frac{1}{n}$, by the transitive property we get that $\frac{1}{n+1} < \frac{\epsilon}{2}$
   By the scratch work: This is equivalent to exactly what we wanted:
   $1 - \frac{2}{n+1} > 1 - \epsilon$. Now we are done

   😊

Note:-

Scratch Question for proof

Note that the scratch is not enough by itself. We need to incorporate a formal proof as shown above. Recommended to do 3 or 4 for practice.

Example 5.1.3 (Exercise 3)

Let $A = [0,1) \bigcap \mathbb{Q}$. We will not do the easy parts. 0 is a minimum, so that one is fine. We guess that $\sup A = 1$. It is clearly an upper bound because of the interval. To show that it is the least upper bound, we need to show that for all $\epsilon > 0$, there is an element $a \in A$ such that $a > 1 - \epsilon$. For this question we need to be a little more creative:

Note:-

Scratch

We look for $a \in [0,1) \bigcap \mathbb{Q}$ such that $a > 1 - \epsilon$. We want a rational number less than 1, such that $a > 1 - \epsilon$ is true. It will be better if we restrict our search to a specific type of $\mathbb{Q}$. Restrict a to be of the form: $a = 1 - \frac{1}{n} \in \mathbb{Q}$, such that $1 - \frac{1}{n} > 1 - \epsilon$. Do this for practice. Possible by the Archimedean Property.

Example 5.1.4 (The Existence of Square Roots From the textbook)

Theorem 5.1.2

There exists a real number $\alpha \in \mathbb{R}$ satisfying $\alpha^2 = 2$.

Proof: Consider the set $T = \{t \in \mathbb{R} : t^2 < 2\}$ and set $\alpha = \sup T$. We are going to prove that $\alpha^2 = 2$ by ruling out possibilities $\alpha^2 < 2$ and $\alpha^2 > 2$. Keep in mind that there are two parts to the definition of $\sup T$ and they will both be important. (This always happens when a supremum is used in an argument). The strategy is to demonstrate that $\alpha^2 < 2$ violates the fact that α is an upper bound for T, and that $\alpha^2 > 2$ violates the fact that it is the least upper bound. Let's first see what happens if we assume $\alpha^2 < 2$. In search of an element of T that is larger than α, write:

$$(\alpha + \tfrac{1}{n})^2 = \alpha^2 + \tfrac{2\alpha}{n} + \tfrac{1}{n^2}$$
$$< \alpha^2 + \tfrac{2\alpha}{n} + \tfrac{1}{n}$$
$$= \alpha^2 + \tfrac{2\alpha+1}{n}.$$

But now assuming $\alpha^2 < 2$ gives us a little space in which to fit the $\frac{(2\alpha+1)}{n}$ term and keep the total less than 2. Specifically, choose $n_0 \in \mathbb{N}$ large enough so that:

$$\tfrac{1}{n_0} < \tfrac{2-\alpha^2}{2\alpha+1}.$$

This implies that $\frac{(2\alpha+1)}{n_0} < 2 - \alpha^2$, and consequently that:

$$(\alpha + \tfrac{1}{n_0})^2 < \alpha^2 + (2 - \alpha^2) = 2.$$

Thus, $\frac{\alpha+1}{n_0} \in T$, contradicting the fact that α is an upper bound for T. We conclude that $\alpha^2 < 2$ cannot happen. Now, what about the case $\alpha^2 > 2$? This time, write:

$$(1 - \tfrac{1}{n})^2 = \alpha^2 - \tfrac{2\alpha}{n} + \tfrac{1}{n^2}$$
$$> \alpha^2 - \tfrac{2\alpha}{n}.$$

The remainder of this argument is requested in exercise 1.4.7.

😊

Chapter 6

Tuesday September 10th

Quiz tomorrow. Given a set compute the supremum and infimum. One of the harder quizzes.

Office Hours Today. Not tomorrow or Thursday. From 5:00-6:00PM.

6.1 Reminder

Definition 6.1.1: Axiom of Completeness

Every $A \neq \emptyset$ bounded above subset of the $\mathbb{R}$, has a supremum (least upper bound).

Definition 6.1.2: Archimedean Property

$\mathbb{N}$ is not bounded above. The equivalent version that we use:
For all $\epsilon > 0$, there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$

6.2 Finish The Proof from last time

Theorem 6.2.1 The Square Root of 2 is in the reals
There exists a real number $\alpha^2 = 2$.

Proof: Consider $T = \{x \in \mathbb{R} : x^2 < 2\}$

$$\alpha = \sup T > 0$$

Note:-
Last Time

Last time we showed that $\alpha^2 < 2$ gave us a contradiction α being an upper bound of the set T.
We can deduce now that $\alpha^2 \geqslant 2$.
Suppose that $\alpha^2 > 2$ is true. We'll show that this contradicts α being the least upper bound of T.
We'll find a $n \in \mathbb{N}$ such that $\alpha - \frac{1}{n}$ is an upper bound of T. This is enough to get us a contradiction because this number is strictly smaller than α, which means α is not an upper bound.

Note:-
Scratch Work

We seek $n \in \mathbb{N}$ such that, $\alpha - \frac{1}{n}$ is an upper bound of T. Enough to make $(\alpha - \frac{1}{n})^2 > 2$. Try to make it look like
A.P. this is equivalent to asking $\alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} > 2$
Since: $\alpha^2 - \frac{2}{\alpha}n + \frac{1}{n^2} > \alpha^2 - \frac{2\alpha}{n}$,
It is enough to make $\alpha^2 - \frac{2\alpha}{n} > 2$

This is equivalent to asking $\frac{2\alpha}{n} < \alpha^2 - 2$

$= \frac{1}{n}\frac{\alpha^2-2}{2\alpha}$. We can make this claim because of our assumption is that $\alpha^2 = 2$. We set the right hand side to be equal to ϵ.

$\epsilon = \frac{\alpha^2-2}{2\alpha} > 0$. Continue from Enough to make. We look for $n \in \mathbb{N}$ such that $(\alpha - \frac{1}{n})^2 > 2$. Set $\epsilon = \frac{\alpha^2-2}{2\alpha}$, since we assumed that $\alpha^2 > 2$ and we know that $\alpha > 0$, it follows that $\epsilon > 0$.(We can only do this if whatever is on the right hand side is positive.) By the A.P. there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon \leftrightarrow \frac{1}{n} < \frac{\alpha^2-2}{2\alpha}$. From our scratch work we get $(\alpha - \frac{1}{n})^2 > 2$. Thus $\alpha - \frac{1}{n}$ is an upper bound of T. This is a contradiction because α was assumed to be the least upper bound of T.

Note:-

Question

We want $n \in \mathbb{N}$ such that $\alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} > 2$.

If instead we find $n \in \mathbb{N}$ such that $\alpha^2 - \frac{2\alpha}{n} > 2$, then since $\alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} > \alpha^2 - \frac{2\alpha}{n}$ by the transitive property we can get our conclusion.

😊

In HW3 You will show $S = \{x \in \mathbb{Q} : x^2 < 2\}$ and show that the supremum of this set is again $\sup S = \sqrt{2}$.

6.3 Same Argument for $\sqrt{p}$

There was nothing special about $\sqrt{2}$, a similar argument will imply that $\sqrt{p}$ exists in $\mathbb{R}$ for every $p_i \in \mathbb{N}$.

Note:-

We have constructed all square roots

6.4 Third Player

Definition will be formal later:

Definition 6.4.1: Density

The set $\mathbb{Q}$ is dense in $\mathbb{R}$

Theorem 6.4.1 Integer has a maximum element

Every non empty bounded above subset of the $\mathbb{Z}$, has a maximum element.

Note:-

Why this is true

There is exactly one number separating every other integer. Whole numbers have fixed gaps between them.

Note:-

Maximum and Supremum

A supremum is defined as a maximum.

Note:-

Axiom of Completeness

Axiom of completeness gives us maximums. Maximums are not always in the set however.

Proof: Let $B \neq \emptyset, B \subseteq \mathbb{Z}$, and B bounded above. The Axiom of Completeness guarantees that the supremum in B exists, $\sup B = s$ exists. We want to show that $s \in B$.

s is the least upper bound of B implies that $s - \frac{1}{2}$ is not an upper bound of B, because it is a striclty smaller number than s. Not an upper bound means that there exists an element $m \in B$ such that $m > s - \frac{1}{2}$.
This is equal to: $s < \frac{1}{2} + m < m + 1$. This implies that $m + 1$ is an upper bound of B. It must not be an element of our set by our assumption. $m + 1 > s \to m + 1 \notin B$.
$m \leqslant s < m + 1$, where $m \in B$ and $m + 1 \notin B$. Since B only contains integers, m must be the last element within our set. Or m must be the maximum.

😄

Note:-

We will show the same property for HW

Recreate the above argument

6.5 Formal Definition of Density

Definition 6.5.1: Definition 2 of Density

Suppose $D \subseteq \mathbb{R}$, we say that D is dense in $\mathbb{R}$ if: for all $a, b \in \mathbb{R}$ with $a < b$, there exists an element of the set $d \in \mathbb{R}$ such that $a < d < b$. (There exists an element of the set that is strictly between them). Between any two real numbers, we can find a number in our special set D that is between them.

Theorem 6.5.1 $\mathbb{Q}$ is dense in $\mathbb{R}$

$\mathbb{Q}$ is dense in $\mathbb{R}$, for any real numbers $a, b \in \mathbb{R}$ with $a < b$, there exists a $q \in \mathbb{Q}$ such that $a < q < b$. In other words between any two real numbers there is a rational number.

Note:-

Scratch Idea

$a < b \to b - a > 0$. Let $\epsilon = b - a$. By the Archimedian Principle there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < b - a$. We have $\frac{1}{n} \in \mathbb{R}$.

Note:-

Scratch Idea # 2

Midpoint Argument. The problem is that a, b don't have to be rational themselves.

Note:-

Quizzes and Tests

Might be asked to state a theorem or a definition.

Proof: Let $a, b \in \mathbb{R}$ with $a < b$. We want to find a $q = \frac{m}{n} \in \mathbb{Q}$ such that $m, n \in \mathbb{Z}$ with $n \neq 0$, that is strictly between $a < \frac{m}{n} < b$.
If $a < 0 < b$. This is the easy case because we can take $q = 0$.
Assume that $0 \leqslant a < b$. This is the main case. Step 1 is to construct $n \in \mathbb{N}$:
$a < b \to b - a > 0$. Set $\epsilon = b - a$. By the A.P. there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < b - a$. Let this be relation one.
Step 2 will be to construct the numerator:
Consider the set $B = \{m \in \mathbb{Z} : \frac{m}{n_0} < b\}$. First observation is that b is non empty. 0 is an element here because $b > 0$. b is bounded above. This follows by construction, if $m \in B$ then $\frac{m}{m_0} < b \to m < n_0 b \to n_0 b$ is an upper bound of B. The last observation is that B only contains integers $B \subseteq \mathbb{Z}$.
(We know that bounded above subsets of the integers have a maximum from our previous theorem:)
We know that every non empty bounded above subset of $\mathbb{Z}$ has a maximum: B has a maximum. Let m_0 be the maximum element of B: $m_0 \in B$.
$m_0 \in B \to \frac{m_0}{n_0} < b$. The last step is to show that $\frac{m_0}{n_0} > a$.
m_0 is the maximum of B implies that the next integer is not in B: $m_0 + 1 \notin B \to \frac{m_0+1}{n_0} \leqslant b$

$$\frac{1}{n_0} \geqslant b - \frac{m_0}{n_0}$$
$$b - a\frac{1}{n_0}$$
By the transitive property we get that $b - a > b - \frac{m_0}{n_0}$
$$\frac{m_0}{n_0} > a.$$

😊

Note:-

HW3

$S = \{x \in \mathbb{Q} : x^2 < 2\}$, recreate the proof using real numbers and recreate it for density.

6.6 True False

1. The set $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ of irrational numbers is also dense in $\mathbb{R}$.
   This is True.

   Proof: Proof of all of the irrationals being dense. Let $a, b \in \mathbb{R}$ such that $a < b$. We want to find $r \in \mathbb{R} - \mathbb{Q}$ that is strictly in between them: $a < r < b$. we have $a < b \longrightarrow a + \sqrt{2} < b + \sqrt{2}$ to both sides. Now by the density of $\mathbb{Q}$, in the $\mathbb{R}$, there exists a $q \in \mathbb{Q}$ such that $a + \sqrt{2} < q < b + \sqrt{2}$. We can subtract $\sqrt{2}$ and we are left with $a < q - \sqrt{2} < b$. Let $r = q - \sqrt{2}$. We claim that this is really an irrational number. $r \in \mathbb{I}$. Subtracting an irrational number from a rational number is still irrational? 😊

2. Given $a, b \in \mathbb{R}$ with $a < b$ there exists infinitely many rational numbers between them.

   Proof: Start with $a < b$ density of $\mathbb{Q} \in \mathbb{R}$ gives $q \in \mathbb{Q}$ with $a < q_1 < b$. But Density also applies for $a < q_1$. There will be another $a < q_2 < q_1$, and so on. 😊

6.7 Setup something for next time

Definition 6.7.1: Nested Intervals

Suppose that for each $n \in \mathbb{N}$, we are given a closed interval: $I_n = [a_n, b_n]$. A closed interval means $I_n = \{x \in \mathbb{R} : a_n \leqslant x \leqslant b_n\}$. We will say that I_n are nested if $I_n \supseteq I_{n+1}$ for all $n \in \mathbb{N}$.

Note:-

Observation

If $\{I_n = [a_n, b_n] : n \geqslant 1\}$ is a nested sequence of closed intervals then: $a_1 leq a_2 \leqslant \ldots \leqslant a_n \leqslant b_n \leqslant b_{n-1} \leqslant \ldots \leqslant b_1$, is true for all $n \in \mathbb{N}$.

Chapter 7

Thursday September 12th

7.1 Reminders

Definition 7.1.1: Archimedian Property

For any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$

Definition 7.1.2: Density

$\mathbb{Q}$ is dense in $\mathbb{R}$, i.e., for any $a, b \in \mathbb{R}$ such that $a < b$, there is $q \in \mathbb{Q}$ such that $a < q < b$. For every interval of real numbers you have a rational number

7.2 Today

Definition 7.2.1: Nested Intervals

Intervals where one is inside the other. A sequence of these intervals. Let $I_1 \supseteq I_2 \supseteq I_e \ldots \supseteq I_n$, be a nested sequence of **closed** intervals. That is $I_n = [a_n, b_n]$, $I_n \supseteq I_{n+1}, n = 1, 2, \ldots,$.

Each interval exists inside of the interval to its left. It follows from this that:
$a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n \leqslant a_{n+1} \leqslant \ldots \leqslant b_{n+1} \leqslant b_n \leqslant \ldots \leqslant b_1$.

Theorem 7.2.1 Nested Interval Problem

Assume $\{I_n = [a_n, b_n]\}_{n \geqslant 1}$ is a nested sequence of closed intervals.

We can conclude that the intersection of them is not empty:
$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$, There is at least one point that is in all of these intervals

Proof: We will show that $x \in \bigcap_{n=1}^{\infty} I_n$, that is $x \in I_n$ for all $n \geqslant 1$.

Consider the set $A = \{a_n : n \geqslant 1\}$, set of all of the left end points.

1. We know that $A \neq \emptyset$, meaning $a_1 \in A$.

2. We also know that A is bounded above. This follows from the fact each of the a_i has a corresponding b_i that bounds it above.

   Proof: Claim that b_m is an upper bound of A for any $m \in \mathbb{N}$

That is $a_n \leqslant b_m$ for any $n, m \in \mathbb{N}$.This follows from the picture we drew, i.e., definition of our nested sequence of closed intervals.

Take cases. If $n < m$, then that means $I_m \subseteq I_n$, this means that $a_n \leqslant a_m \leqslant b_m \leqslant b_n$. We only need to show $a_n \leqslant b_m$. Do the other cases on your own.

😊

Note:-

Conclusion 1

From 1 and 2 above, we know that sup$A$ exists. This follows from the Axiom of Completeness. $x :=$ sup$A$ exists in $\mathbb{R}$

Note:-

Conclusion 2

$a_n \leqslant x \leqslant b_n$ for all n. Indeed, $a_n \leqslant x$, since x =sup$A$ =sup$\{a_n : n \geqslant 1\}$.
Also since b_n is an upper bound of A, (because of the claim above), then $x =$ sup$A \leqslant b_n$ for any fixed n. This concludes the proof, because for any n I found an x that is in all of these intervals.

😊

Note:-

What was important in our assumptions

The main thing that we used: The Axiom of Completeness, what we just said might not be necessarily true for rational numbers.

Another thing that is important is that we took a nested sequence of closed intervals.

Question 1

Would the conclusion hold if we had a nested sequence of open, (half-open), intervals?

Change the intervals to open and non empty.

Solution: No. For example:

Let $I_n = (0, \frac{1}{n}), n \in \mathbb{N}$

Claim: $\bigcap_{n=1}^{\infty} I_n = \bigcap_{n=1}^{\infty}(0, \frac{1}{n}) = \emptyset$

Indeed if $\bigcap_{n=1}^{\infty}(0, \frac{1}{n}) \neq \emptyset$, then there is $x \in (0, \frac{1}{n})$ for all n. That is $0 < x < \frac{1}{n}$ for all n.

Real number that is positive and less than $\frac{1}{x}$ for all n. This contradicts the Archimedian Property, as defined above.

This is because the Archimedian Property is: Given $x > 0$, there is $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.

7.3 Infinities

Note:-

We know that $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are infinite. The Natural numbers, integers, rationals, and reals are infinite.

Question 2

Is there one of them that is more infinite than the others?

Solution: Yes?

Definition 7.3.1: One to One: Injective Function

A function $f : A \to B$ is called one to one if for any $a_1, a_2 \in A$ with $a_1 \neq a_2$, we have $f(a_1) \neq f(a_2)$.

Equivalently, f is 1-1 if $(f(a_1) = f(a_2)) \to a_1 = a_2$.

Definition 7.3.2: Onto: Surjective Function

A function $f : A \to B$ is called onto if the range equals the codomain, every $b \in B$ has a corresponding $a \in A$ such that $f(a) = b$.

Definition 7.3.3: Bijective Function

If $f : A \to B$ is both 1-1 and onto, it is called a bijective function

Definition 7.3.4: Cardinality

Let A, B be two sets, we say that A and B have the same cardinality if there is a bijection from A to B

In that case we write, $A \sim B$.

If two sets have the same cardinality, then they should have exactly the same number of elements.

Note:-

What Does this do?

You can identify elements of the first set with elements of the second set in a one to one fashion. Exactly every element in A is mapped to exactly one element in B.

Question 3

How do we know if a set is infinite?

Solution: Define the natural numbers to be infinite and show a bijection between the set in question and the naturals.

Definition 7.3.5: Finite Set

We say that a set A is finite if $A = \emptyset$, or $A \sim \{1, \ldots, n\}$ for some (fixed) $n \in \mathbb{N}$. In the first case the cardinality is 0, in the second case the cardinality is n.

Definition 7.3.6: Infinite Set

We say that a set is infinite if it is not finite.

Example 7.3.1 (Example 1)

If A and B are finite then $A \sim B$ if and only if A and B have the same number of elements.

Proof: If the two sets have the same cardinality, it is harder to show that the integer has to be the same for both A and B. The other direction is constructing a bijection between the two sets. Map the n_{th} element of the first set to the n_{th} element of the second for $n \in \mathbb{N}$

😊

Example 7.3.2 (Example 2)
$\mathbb{N}$ is infinite.

Definition 7.3.7: Countability

A set A is called countable if $A \sim \mathbb{N}$.

Only taking infinite sets are countable. This is a convention.

That is if there is a bijection $f : \mathbb{N} \to A$.

Definition 7.3.8: Uncountability

An infinite set that is not countable is called uncountable.

Question 4

If $\mathbb{N}$ is countable and $\mathbb{R}$ is uncountable is there anything that is bigger than $\mathbb{N}$ but less than $\mathbb{R}$?

Solution: There is no way to prove if such a set exists.

Note:-

If A is countable, we can write its elements as a sequence. That is we can index the elements by the natural numbers.

This follows from the fact that we can construct a bijection between A and $\mathbb{N}$.

$f : \mathbb{N} \to A$, then $1 \to f(1) := a_1$, $2 \to f(2) := a_1$, and therefore $A = \{a_1, a_2, \ldots, a_n\}$.

Example 7.3.3
$\mathbb{N}$ is countable, indeed $f : \mathbb{N} \to \mathbb{N}$, this implies that $f(n) = n$. This is called the identity funciton, and $f(n)$ is 1-1 and onto.

Example 7.3.4
Let $E^+ = \{2, 4, 6, \ldots\}$, these are the positive even numbers. Then E^+ is countable. Indeed, $f : \mathbb{N} \to E^+$, $f(n) = 2n$. This is 1-1 and onto.

7.4 $\mathbb{Q}$ is Countable

Lenma 7.4.1
Assume that $A_n \subseteq \mathbb{R}$ is finite for $n \in \mathbb{N}$, and such that $A_n \bigcap A_m \neq \emptyset$ if $n = m$. (A_n are pairwise disjoint)

Then their union is countable: $\bigcup_{n=1}^{\infty} A_n$ is countable.

Proof: Since A_1 is finite, we can write $A_1 = \{a_{1,1}, a_{1,2}, ..., a_{1n_1}\}$ where n_1 is the cardinality of A. Likewise $A_2 = \{a_{2,1}, a_{2,2}, ..., a_{2,n_2}\}$ and so on. For each A_i we do the same thing, for $i \in \mathbb{N}$.
Then we define a function $f : \mathbb{N} \to \bigcup_{n=1}^{\infty} A_n$:
$f(1) = a_{1,1}, f(2) = a_{1,2}, ..., f(n_1) = a_{1,n_1}$
$f(n_1 + 1) = a_{2,1}, ..., f(n_1 + n_2) = a_{2,n_2}$. And so on. We can conclude that f is a bijection. All the elements of A_1 up to A_n are mapped. Additionally they are mapped to unique values.

😄

Theorem 7.4.1

The set of rational numbers $\mathbb{Q}$ is countable.

Proof: We define $A_1 = \{0\}$ and $A_n = \{\pm\frac{q}{p} :$ where p, q have no common divisor, and $p + q = n\}$.
We define the remaining set for $n \geqslant 2$:
$A_2 = \{-1, -1\}$.
Take portions of natural numbers that add to 3. $A_3 = \{\pm\frac{1}{2}, \pm 2\}$,
$A_4 = \{\pm\frac{1}{3}, \pm 3\}$
$A_5 = \{\pm\frac{1}{4}, \pm\frac{2}{3}, \pm\frac{3}{2}, \pm 4\}$.
And so on ...
Claim that A_n are pairwise disjoint.

😄

Chapter 8

Tuesday September 17th

Note:-
Quizzes: people did well and people did poorly there will be a problem like this one on the test. Or something like it. Perhaps sequences?

Note:-
Thursday October 3rd Midterm. We will have HW until the week before. Then that week there won't be HW but there will be review questions.

Note:-
Won't provide solutions to review Questions.

Note:-
Solutions to HW1 and HW2 are uploaded, and 3 and 4 will be uploaded.

Note:-
HW assignment due this week and next week, and the review assignment questions for the week of the test.

Note:-
Material will be everything up till next Thursday's Lecture.

Note:-
No Quiz this week. Quiz next week.

8.1 Today's Lecture

Note:-
Relatively Hard Pay attention

8.2 Reminders

Definition 8.2.1: Countable

A set $A \subseteq \mathbb{R}$ is countable if there exists $f : \mathbb{N} \to A$, that is $1:1$, onto. The notation for this is $A \sim \mathbb{N}$.

Definition 8.2.2: Uncountable

A set A is uncountable if it is not countable.

Note:-

A is countable $\longrightarrow$ we can write its elements as a sequence: $\{a_1, a_2, \ldots, a_n\}$, meaning we can index them by naturals.

Note:-

$\mathbb{N}, \mathbb{Z}, \mathbb{E}$ are all countable.

Theorem 8.2.1 The Rationals are countable

$\mathbb{Q}$ is countable.

Lemma 8.2.1 Let $A_1, A_2, \ldots, A_n, \ldots$ be finite disjoint sets. Then $A = \bigcup_{n=1}^{\infty} A_n$ is countable.

Proof: Consider the set $A_1 = \{0\}$, For $n \geqslant 2$, $A_n = \{\pm\frac{p}{q} : \frac{p}{q}$ reduced $, p, q \in \mathbb{N}, p + q = n\}$.
Step 1) Each A_n is finite. A_1 is finite. Let $n \geqslant 2$, and we want to show that A_n is finite. There are only finitely many ways we can write $n = p + q$ with $p, q \in \mathbb{N}$. This implies that A_n is finite.

Question 5: Do you have to justify this fact that $n = p + q$?

Solution:

If $n = p + q$ with $p, q \in \mathbb{N}$, then $p, q \in \mathbb{N} \bigcap [1, n]$. This is a finite set which shows that A_n is finite.
Step 2) The A_n's are pairwise disjoint. We want to show that for $n \neq m$, $A_n \bigcap A_m = \emptyset$. Note the special case of $A_1 \bigcap A_n = \emptyset$. This is clear since $0 \notin A_n$ for $n \geqslant 2$. Now consider $n, m \geqslant 2, n \neq m$

Note:-

Proving two sets are disjoint. We always go by contradiction. Showing something is the emptyset, we should go by contradiction.

Suppose by contradiction that $A_n \bigcap A_m \neq \emptyset$. This means that we can find at least one element in there: $x \in A_n \bigcap A_m$. We know that $x \in \mathbb{Q} \longrightarrow x = \frac{l}{k}$ or $-\frac{l}{k}$ for some $l, k \in \mathbb{N}$. Reducing the fraction implies that x has unique expression as $x = \frac{p}{q}$ or $x = -\frac{p}{q}$ with $p, q \in \mathbb{N}$ and $\frac{p}{q}$ reduced. Now $x \in A_n \longrightarrow p + q = m$. Additionally, $x \in A_m = p + q = m$. These two facts that $n = m$ and this is a contradiction.
The lemma above implies that $\bigcap_{n=1}^{\infty} A_n$ is countable.
The last step is to show that $\bigcup_{n=1}^{\infty} A_n = \mathbb{Q}$.
Step 3) We need to show $\subseteq, \supseteq$ but $\bigcup_{n=1}^{\infty} A_n \subseteq \mathbb{Q}$ is clear because each $A_n \subseteq \mathbb{Q}$.
The other direction: Let $x \in \mathbb{Q}$: Special Case: if $x = 0 \longrightarrow x \in A_1$.
If $x \neq 0$, then we can write x uniquely in the form $x = \frac{p}{q}$ or $-\frac{p}{q}$ for some $p, q \in \mathbb{N}$ with p, q reduced. Take $n = p + q$, By definition $x \in A_n$.
Thus $\mathbb{Q} = \bigcup_{n=1}^{\infty} A_n$ is countable.

☺

Note:-

We know that $\mathbb{N}$,$\mathbb{Z}$,$\mathbb{Q}$ are countable. Next we will want to explore things like $\mathbb{R}$, intervals like $(0, 1)$, closed intervals $[a, b]$, and the irrationals.

Theorem 8.2.2 $(-1, 1) \sim \mathbb{R}$. This implies that we can find a one to one and onto function between the two sets.

Note:-

Just for this example we will use calculus.

Proof: We need to construct $1 : 1$, onto function $f : (-1, 1) \longrightarrow \mathbb{R}$.
Take $f : (-1, 1)\mathbb{R}$
$f(x) = \frac{x}{x^2-1}$

Claim that this is $1 : 1$ and onto.

Question 6: How using Calculus can we show that this is a bijection?

Solution:

Take the derivative, if the funciton is strictly increasing or strictly decreasing the function will definitely be one to one.

Check $f : 1 : 1$ using calculus. Use the chain rule:

$f'(x) = \frac{-1-x^2}{(x^2-1)^2}$. This is clearly negative. Calculus tells us that f is strictly decreasing. Meaning if x, y are elements in the domain with $x < y$, this implies that $f(x) > f(y)$. This implies that f is $1 : 1$.

Question 7: How can we use calculus to show that f is onto?

Solution:

Take the limit at 1 and -1. We need to compute the $\lim_{x \to -1^+} = \lim_{x \to -1^+} \frac{x}{x^2-1} = \infty$ and that $\lim_{x_1^-} f(x) = -\infty$.

These two facts imply that the range of $f = (-\infty, \infty) = \mathbb{R}$.

This depends on the fact that f is continuous which implies that the range of $f = \mathbb{R}$.

Note:-

We are encouraged to check these two directions without calculus on our own.

😄

Note:-

For HW4 you will show that $[a, b] \sim (a, b) \sim (a, b] = \mathbb{R}$.

Note:-

What we will show in the next HW:

1. If A, B countable $\longrightarrow A \cup B$ is countable

2. If A countable and $B \subseteq A \longrightarrow B$ is finite or countable.

Theorem 8.2.3 $\mathbb{R}$ is uncountable.

Note:-

Reminder:

Definition 8.2.3: Nested Closed Interval Property

Suppose $I_n = [a_n, b_n], I_1, \supseteq I_2 \supseteq \ldots \supseteq I_n$ is a nested sequence of closed intervals. Then the conclusion is that $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Note:-

We will do 4 proofs throughout the semester sort of similar to this one.

Note:-

To show a set is uncountable, either find a one to one and onto funciton to a set that we already know is uncountable. But we need to establish that one set is uncountable first. To do so we will do this by contradiction.

Proof: Suppose for contadiction, that $\mathbb{R}$ is countable. This means that we can index the elements by the natural numbers, we can write them in sequence. $\mathbb{R} = \{x_1, x_2, \ldots, x_n, \ldots\}$.

Our strategy: We will construct a sequence of nested closed intervals $I_1 \supseteq I_2 \ldots$ such that the intersection is empty: $\bigcap_{n=1}^{\infty} I_n = \emptyset$. This will contradict the nested closed interval property.
We will construct the intervals I_n inductively so that we have the following property: the nth element of $\mathbb{R}$ is not in the interval of I_n: $x_n \notin I_n$.
Step 1): Construct $I_1 = [a_1, b_1]$ such that $x_1 \notin I_1$. x_1 is a real number the Archimedian property guarantees that we can find a natural number that is strictly larger. Take $I_1 = [n, n+1]$. Since $n > x_1$ we know that $x_1 \notin I_1$.
Step 2): Construct $I_2 \subseteq I_1$ such that $x_2 \notin I_2$.
Case 1: If x_2 is not in I_1 then take $I_2 = I_1$. Works.
Case 2: Suppose that x_2 is an element of I_1. Split I_1 into two sub intervals: $[n, y], [y, n+1]$ in a way so that x_2 is in only one of the two sub intervals.
Chose I_2 to be the sub interval to be the sub interval not containing x_2.
Continue inductively. Having constructed closed intervals $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n$ with $x_i \notin I_i$ for all $i \leqslant n$, we construct the next $I_{n+1} \subseteq I_n$ as follows:

1. If $x_{n+1} \notin I_n$ then set $I_n = I_{n+1}$
2. If $x_{n+1} \in I_n \longrightarrow$ split I_n into 2 sub intervals and set $I_{n+1} =$ the one not containing x_{n+1}

This implies that we have built a nested sequence of closed intervals $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n$ with the property that $x_n \notin I_n$ for all $n \in \mathbb{N}$.
Claim is that $\bigcap_{n=1}^{\infty} I_n = \emptyset$. Suppose that $\neq \emptyset$, then there exists $x \in \mathbb{R}$ with $x \in I_n$ for all $n \in \mathbb{N}$, but initially we assumed that $\mathbb{R} = \{x_1, x_2, \ldots, x_n\}$ which implies that $x = x_i$ for some $i \in \mathbb{N}$ by our construction x_i is not in the set I_i which implies that $x \notin \bigcap_{n=1}^{\infty} I_n$ which is a contradiction.
This contradicts our initial assumption that $\mathbb{R}$ is countable. Therefore $\mathbb{R}$ is uncountable.

😊

Question 8

$\mathbb{R} \supseteq \mathbb{Q}$: $\mathbb{R}$ satisfies the axiom of completeness but $\mathbb{Q}$ does not. In fact the axiom of completeness is equivalent to the nested interval property.

Note:-

We know now that $\mathbb{Q}, \mathbb{Z}, \mathbb{N}$ is all countable. We know that the $\mathbb{R}$, and some intervals are uncountable (HW).

Question 9: The set $I = \mathbb{R} - \mathbb{Q}$ is uncountable.

Solution: True. If A, B countable then $A \bigcup B$ is countable.

Proof: If $\mathbb{I}$ were countable then $\mathbb{R} = \mathbb{Q} \bigcup \mathbb{I}$ would also be countable.

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Definition 8.2.4: Reminder one more time Lemma 1

Given $\{A_n\}_{n=1}^{\infty}$ finite disjoint then $\bigcup_{n=1}^{\infty} A_n$ is countable.

Lemma 8.2.2 Strengthen Lemma 1

Let $\{A_n\}_{n=1}^{\infty}$ finite subsets of $\mathbb{R}$, then the union: $A = \bigcup_{n=1}^{\infty} A_n$ is either finite or countable.

Proof: We proved this when the sets are disjoint, now just allow the sets not to be disjoint. If $\bigcup_{n=1}^{\infty} A_n$ is finite. This is ok. Suppose now that $\bigcup_{n=1}^{\infty} A_n$ is infinite, consider the sets $B_1 = A_1$, $B_2 = A_2 \setminus A_1 = \{x \in A_2 : x \notin A_1\}$, we continue in this manner $B_3 = A_3 \setminus (A_1 \bigcup A_2)$ and so forth. We will continue this next time.

😊

Chapter 9

Thursday September 19

9.1 Left-over from Last Time

Note:-
Finish up countability

Note:-
$\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ countable, but $\mathbb{R}, \mathbb{I}, (a,b), [a,b), [a,b]$ uncountable.

Lemma 9.1.1 Didn't Finish From Last Time
Let $A_1, A_2, ..., A_n$ be finite subsets of $\mathbb{R}$, Then $A = \bigcup_{n=1}^{\infty} A_n$ is either finite or countable.

Note:-
We know this is true if A_i is pariwise disjoint.

Proof: If $A = \cup_{n=1}^{\infty} A_n$ is finite we are done, so we will suppose infinite. We consider the sets $B_1 = A_1$, $B_2 = A_2 \setminus A_1$. (The elements in A_2 that are not in A_1). We proceed in this manner:
$B_3 = A_3 \setminus (A_1 \cup A_2)$. More generally,
For $n \geqslant 2$:$B_n = A_n \setminus (A_1 \cup ... \cup A_{n-1})$.
Now we want to verify the following. Each B_n is finite. For each $n \geqslant 1$, $B_n \subseteq A_n$. Since A_n is finite we know that B_n is also finite.
B_n are pairwise disjoint. We want to show that for $n \neq m, B_n \cap B_m = \emptyset$. Let $n \neq m$ without loss of generality assume that $n < m$. Then the set $B_m = A_m \setminus (\cup_{i=1}^{m-1} A_i)$ and because $B_n \subseteq A_n \subseteq \cup_{i=1}^{m-1} A_i$ so therefore they are disjoint.
The last thing to show is that $\cup_{n=1}^{\infty} B_n = \cup_{n=1}^{\infty} A_n = A$.
We need to show double inclusion here $\supseteq, \subseteq$. Since $B_n \subseteq A_n$, for all $n \in \mathbb{N}$ then $\cup_{n=1}^{\infty} B_n \subseteq \cup_{n=1}^{\infty} A_n$.
In the other direction, let $x \in \cup_{n=1}^{\infty} A_n \longrightarrow$ this implies that there exists $n \in \mathbb{N}$ such that $x \in A_n$. Now we will pick the smallest n that makes this true.
Let $n_0 \in \mathbb{N}$ be the smallest such that $x \in A_{n_0}$, this means that $x \notin A_i$ for all $i \in \{1, ..., n_0 - 1\}$. This implies that $x \in A_{n_0} \setminus (\cup_{i=1}^{\infty} A_i) = B_n$ by definition. So we are done. This gives us $x \in B_{n_0}$ and therefore $x \in \cup_{n=1}^{\infty} B_n$. Since B_n's are finite and pairwise disjoint the first lemma we showed implies that the union of the $B_n's$ is the same as the union of the $A_n's$ is countable. And this finishes our proof.

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Note:-
If the union is infinite, then it will be countable.

9.2 Summary for countable sets

1. If $A_1, ..., A_n$ are all finite, then their union: $\cup_{n=1}^{\infty} A_n$ is either finite or countable.

2. From HW4, we will show that if $A_1, A_2, \ldots, A_n$ are countable, then their union is countable.
3. Meaning countable unions of countable sets are countable.

Note:-

Next HW, and on the exam I will give you an explicit subset of the real numbers and I will ask you if it is countable or not.

Most of the times you will write it as the first and second bullet point above to explain whether or not it is countable.

Note:-

You will need to use the lemmas and results from the HWs to show these problems.

Chapter 10

Chapter 2: Sequences and Series

Reminder for a sequence:

Definition 10.0.1: Sequence of Real Numbers

A sequence of real numbers is a function $a : \mathbb{N} \longrightarrow \mathbb{R}$. A sequence of real number is a labeling:

$$a_1 = a(1)$$
$$a_2 = a(2)$$

This implies that $\{a_1, a_2, ..., \} \subseteq \mathbb{R}$. a_i can repeat themselves.

Note:-

Notation: $(a_n)_{n\geqslant 1}$ or $\{a_n\}_{n\geqslant 1}$

Definition 10.0.2: Limit Definition of a Sequence

What does it mean as the $\lim_{n\to\infty} a_n = a$?

Let $(a_n)_{n\geqslant 1}$ be a sequence of real numbers. We say that the $\lim_{n\to\infty} a_n = a$ if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, $|a_n - a| < \epsilon$.

Note:-

$\forall$ = is for all or for every.

$\exists$ = there exists.

Definition 10.0.3: Limit Definition in Symbols

$\lim_{n\to\infty} a_n = a$ if: $\forall \epsilon > 0, \exists N \in \mathbb{N} s.t. \forall n \geqslant N, |a_n - a| < \epsilon$

Question 10

What does this weird thing mean?

Solution: Intuition tells us that as the sequence gets larger thwe approach a. Imagine the number number line and approaching a on both sides.

ϵ is really the distance from the limit.

N says that after some point all of the n will be inside the distance from a and ϵ. Then after some point we change to a smaller than before ϵ, and we can find a larger capital N that will be in this range.

Note:-

Given $\epsilon > 0$ we find a step N after which all of the terms of the sequence a_n are in $< \epsilon$ distance from $a = \lim_{n\to\infty} a_n$.

Note:-

$$|a_n - a| < \epsilon \leftrightarrow -\epsilon < a_n - a < \epsilon$$
$$a - \epsilon < a_n < a + \epsilon$$

Definition 10.0.4: ϵ Neighborhood

For $\epsilon > 0, a \in \mathbb{R}$ the ϵ neighborhood of a is $V_\epsilon(a) = (a - \epsilon, a + \epsilon)$.

Definition 10.0.5: Limit Definition with Neighborhood

$\lim_{n\to\infty} a_n = a$ if $\forall \epsilon > 0 \exists N \in \mathbb{N} s.t. \forall n \geqslant N, a_n \in (a - \epsilon, a + \epsilon)$ for $a_n \in V_\epsilon(a)$.

Example 10.0.1 (Example 1)

Consider the sequence $a_n = \frac{(-1)^n}{n}$. The $\lim_{n\to\infty} a_n = 0$. We already know that this limit is 0.

Proof: Start with Scratch Work. Most of the Time Prof. Uses Scratch work to find the right ϵ. Must do formal afterwards.

We want to show $\forall \epsilon > 0 \exists N \in \mathbb{N} s.t.$ if $n \geqslant N$ then $|a_n - a| < \epsilon$.
Whenever we want to show for all start the proof with Let.
Let $\epsilon > 0$. This means that ϵ is arbitrary, meaning we have no control over it, but it is fixed.
Start it the correct way. If you use the definition, and say let on the exam you will get partial credit.

Note:-

Scratch:
Let $\epsilon > 0$: We will try and simplify the $|a_n - a| < \epsilon$.

We assume that $a = 0$ because we already know that it is zero.

$$|a_n - a| = |\frac{(-1)^n}{n}| = \frac{1}{n}$$

We hope to make $\frac{1}{n} < \epsilon$. The Archimedian Property implies that $\exists N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. A.P. produces 1 N. Then if $n \geqslant N$ then $\frac{1}{n} \leqslant \frac{1}{N} < \epsilon$ by transitivity.
The A.P. produces N in the definition of a limit.

Formal:
Let $\epsilon > 0$. Note that $|a_n - 0| = |\frac{(-1)^n}{n}| = \frac{1}{n}$. By the Archimedian Property $\exists N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then if $n \geqslant N \to \frac{1}{n} \leqslant \frac{1}{N}$. By the transitive property this implies that $\frac{1}{n} < \epsilon$. Which is the same as saying $\forall n \geqslant N$, $|\frac{(-1)^n}{n}| < \epsilon$.

😊

Example 10.0.2 (Example 2)

Prove that $\lim_{n\to\infty} \frac{3n-2}{n+1} = 3$. We will prove that this limit is equal to 3.

$a_n = \frac{3n-2}{n+1}$ and $a = 3$. As before we want to show that $\forall \epsilon > 0 \exists N \in \mathbb{N} s.t.$ if $n \geqslant N$ then $|a_n - a| < \epsilon$.

Let $\epsilon > 0$:

Note:-

Scratch:

$$\left|\frac{3n-2}{n+1}-3\right|$$
$$\left|\frac{3n-2-3n-3}{n+1}\right| = \left|\frac{-5}{n+1}\right|$$
$$\frac{5}{n+1}$$

Here we want $\frac{5}{n+1} < \epsilon$. Similarly to what we were doing with $\sup A$. We will set $\epsilon' = \frac{\epsilon}{5}$ and then by Archimedian Property, it is possible to make $\frac{1}{n} < \frac{\epsilon}{5}$.

Proof: Let $\epsilon > 0$. Set $\epsilon' = \frac{\epsilon}{5} > 0$. By the Archimedian Property $\exists N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon'$ then for all $n \geqslant N$ we get that $\frac{1}{n+1} < \frac{1}{N} < \frac{\epsilon}{5}$. By our scratch work this implies that for all $n \geqslant N$:

$$\left|\frac{3n-2}{n+1}-3\right| < \epsilon$$

😄

Note:-

With 100% guarantee this will be on the midterm. 15% safety points on this. Next Wednesday there will be a quiz to prepare this. Write down similar problems to the ones we just did.

Note:-

Reconsider Example 1

Example 10.0.3 (Example 1 going back)

$a_n = \frac{(-1)^n}{n}$. $\epsilon > 0$: This is the challenge and $N \in \mathbb{N}$ is the step. I give you a challenge, and I tell you can you bring the sequence a_n 0.1 closer to 0??

In this case I give you $\epsilon = 0.1$ and we are looking for $\left|\frac{(-1)^n}{n}\right| < 0.1$ and this is true after $N = 11$. But now I give you a new challenge.

Can you bring a_n 0.00001 close to zero. Yes 1000001. This is optimal but it doesn't have to be. After a big step all of them will fall into this pattern.

This should show that ϵ should be thought of as a number smaller and smaller according to what they like. You can not chose it. Given the ϵ that someone provides, you produce a N. In general N grows inversely to ϵ.

On the other hand if we have proved that the $\lim_{n\to\infty} a_n = a$, then we can apply the ϵN definition for any $\epsilon > 0$ that we like.

Question 11

What does it mean that $\lim_{n\to\infty} a_n \neq a$?

Solution: Negation of our definition:

Class Solution:

$\lim_{n\to\infty} a_n \neq a$ means

$\exists \epsilon > 0 \forall N \in \mathbb{N} \exists n \geqslant N$ with $|a_n - a| \geqslant \epsilon$

Chapter 11

Tuesday September 24

11.1 Announcements

Note:-
Midterm Next Thursday

Note:-
Review Questions Uploaded Tomorrow

Note:-
Quiz 3 Tomorrow

Of the form: Prove that the limit of a sequence is: ...

Note:-
Questions like this will be about $\frac{1}{4}$ of your grade

Note:-
MCLC ~ make an appointment for Basic Real Tutoring

Note:-
Material for the Midterm is up to today's Lecture (including)

11.2 Limits of Sequences

Definition 11.2.1: Sequence

Let $(a_n)_{n \geqslant 1}$ be a sequence of real numbers.

$\lim_{n \to \infty} a_n = a$ means:

$\forall \epsilon > 0 \exists N \in \mathbb{N}$ such that if $n \geqslant N$ then $|a_n - a| < \epsilon$.

Example 11.2.1
$a_n = \{\frac{3n-2}{n-10}, n > 22 \wedge 5 \text{ when } n \leqslant 22\}$

We think that the limit of this is 3.

The limit behavior has only to do with the final terms.

Want to show $\forall\epsilon > 0 \exists N \in \mathbb{N}$ such that if $n \geqslant N$ then $|a_n - \epsilon| < \epsilon$. This is our Goal.

Note:-

Scratch Work

Let $\epsilon > 0$ (ϵ is arbitrary but fixed). We can assume that $N \geqslant 23$. Then for $n \geqslant N$, we have $a_n = \frac{3n-2}{n-10}$.

Simplify as much as possible and use the Arhimedian.

$$|a_n - 3| = |\tfrac{3n-2}{n-10} - 3| = \tfrac{28}{n-10}$$
$$\text{We want: } \tfrac{28}{n-10} < \epsilon$$
$$\tfrac{1}{n-10} < \tfrac{\epsilon}{28} = \epsilon' > 0$$

By the Archimedian Property we can find $\frac{1}{N} < \frac{\epsilon}{28}$. We can find $N_0 \in \mathbb{N}$ such that $\frac{1}{N_0} < \frac{\epsilon}{28}$. Want to make N_0 look like $n - 10$.

Take $N = N_0 + 10$. Then $N - 10 = N_0$, so $\frac{1}{N-10} < \frac{\epsilon}{28}$. If $n \geqslant \mathbb{N} \to n - 10 \geqslant N - 10 \to \frac{1}{n-10} \leqslant \frac{1}{N-10} = \frac{1}{N_0} < \frac{\epsilon}{28}$

Note:-

Begin Formal

Proof: Let $\epsilon > 0$, by the Archimedian Property, for $\epsilon' = \frac{\epsilon}{28} > 0$, there exists $N_0 \in \mathbb{N}$ such that $\frac{1}{N_0} < \frac{\epsilon}{28}$. Take $N = max\{23, N_0 + 10\}$. This maximum exists, becuase we are only taking the maximum between two numbers.

Then: if $n \geqslant N$, we have the following two things:

1. $n \geqslant N \to n \geqslant 23$, and therefore $a_n = \frac{3n-2}{n-10}$
2. $n \geqslant N \geqslant N_0 + 10 \to \frac{1}{n-10} \leqslant \frac{1}{N_0} \leqslant \frac{\epsilon}{28}$, by our scratch this gives us $|a_n - 3| < \epsilon$ and we can conclude.

☺

Note:-

The TA will do something like this tomorrow

Question 12

Why was N_0 created? ***Solution:*** N_0 was created to shift the $n - 10$

Question 13

What if the function wasn't piecewise?

Solution: Function wouldn't be defined as well defined. Question would be different.

Note:-

For piecewise, only the final terms matter. The other point is that sometimes we will have to take a capital N that is a maximum.

Example 11.2.2 (Practice)

Take $a_n = \{\frac{1}{n^2}$ when n is even $\wedge \frac{3}{n+1}$ when n is odd$\}$

Solution: To prove that $\lim_{n\to\infty} \frac{1}{n^2} = 0$

Do two proofs. Get N_1, then do the proof of the second one to take N_2 and take the maximum of the two. Prove $\lim_{n\to\infty} \frac{3}{n} = 0$ and get $N_2 \in \mathbb{N}$

For a_n : we take $N = max\{N_1, N_2\}$ or $N = N_1 + N_2$.

Note:-

Say that we want to show that $\lim_{n\to\infty} a_n = a$.

If we find $N \in \mathbb{N}$ for which whenever $n \geqslant N$ it follows $|a_n - a| < \epsilon$,

Then: any $N' \geqslant N$ will work too.

We don't care to find N optimal. Anything that is bigger or equal than both the numbers will do.

11.3 Negation

Definition 11.3.1: Negation of Limit

$\lim_{n\to\infty} a_n \neq a$ means:

Note:-

Flip the quantifiers and keep the order the same. It is very often that we put the such that in the wrong spot.

Negated Version:

$\exists \epsilon > 0$ such that no N works
We want to find an epsilon so that no matter which N we take, the statement will not be true
$\exists \epsilon > 0$ such that $\forall N \in \mathbb{N}$
[if $n \geqslant N \longrightarrow |a_n - a| < \epsilon$] is Not true. Meaning:
$\exists n \geqslant \mathbb{N}$ for which $|a_n - a| \geqslant \epsilon$.

Note:-

Complete Negation

$\lim_{n\to\infty} a_n \neq a$ if:
$\exists \epsilon > 0$ such that $\forall N \in \mathbb{N} \exists n \geqslant N,$ for which $|a_n - a| \geqslant \epsilon$

Example 11.3.1 (Prove that the Limit is not equal to 1)

$\lim_{n\to\infty} (-1)^n \neq 1$

Take $\epsilon = \frac{1}{2} > 0$

Let $N \in \mathbb{N}$:

Question 14

First step is to pick an epsilon, follow the logical statement.

Solution: The order of the quantifiers really matter, so try and approach the problem without changing the quantifiers. Fix the ϵ first.

Take $n = 2N + 1$. This guy is bigger than N and odd. This implies that $a_n = -1$.

Now we compute that happens when we take $|a_n - 1| = |-1-1| = 2$. This is bigger than $\frac{1}{2}$.

No matter which N you give me I can find a term larger than it is which is further than ϵ away from the limit.

Note:-

When I ask you guys to write down the negation, most of you wrote:

$\exists \epsilon > 0$ such that $\forall N \in \mathbb{N}$ if $n \geqslant N$, then $|a_n - a| \geqslant \epsilon$.

If this were the correct negation then $(-1)^n$ would satisfy it.

Definition 11.3.2: Set is Bounded

Let $(x_n)_{n\geqslant 1}$ be a sequence. We say that (x_n) is bounded if $\{x_n : n \in \mathbb{N}\}$ is bounded both above and below.

Example 11.3.2

$x_n = (-1)^n \rightarrow x_n$ is bounded. For example 1 is an upper bound and -2 is a lower bound.

Theorem 11.3.1

Let $(x_n)_{n\geqslant 1}$ be a sequence. If (x_n) converges, meaning $\lim_{n\to\infty} x_n$ exists. Then (x_n) is bounded.

Proof: We know that x_n has a limit. Suppose that $\lim_{n\to\infty} x_n = x$, for some $x \in \mathbb{R}$.
This implies that $\forall \epsilon > 0 \exists N \in \mathbb{N}$ such that if $n \geqslant N$ then $|x_n - x| < \epsilon$. We know this as a result of our problem definition.
We can apply it for any $\epsilon > 0$ of our liking. In particular for $\epsilon = 1$:
$\exists N \in \mathbb{N}$ such that if $n \geqslant N$, then $|x_n - x| < 1$.
Therefore for $n \geqslant N$ we have $-1 < x_n - x < 1$.
$\rightarrow x - 1 < x_n < x + 1, \forall n \geqslant N$.
This is all as a result of the question. This says that after the n_{th} term everything is bounded between $x - 1$ and $x + 1$. We can not set these to be lower bounds just yet however.
Set $M = \max\{x_1, x_2, \ldots, x_{n-1}, x + 1\}$
Set $m = \min\{x_1, x_2, \ldots, x_{n-1}, x - 1\}$
Claim that M is an upper bound for (x_n) and m is a lower bound for (x_n).
We will only show the M. Let $x_n \leqslant M$ for all $n \in \mathbb{N}$. We will verify this. We have 2 cases.
Case 1: If $n \geqslant N$ then we have established that $x_n < x + 1 \leqslant \max\{x_1, x_2, \ldots, x_{n-1}, x + 1\}$, because it was included in the set defining M.
Case 2: If $n < N$ then x_n is one of the terms $x_1, x_2, \ldots, x_{n-1}$. This implies that $x_n \leqslant \max\{x_1, x_2, \ldots, x_{n-1}, x - 1\}$.
☺

Note:-

This theorem says that convergent implies bounded.

Question 15

Is the converse true?

Solution: The converse is not true, take $\lim_{n\to\infty}(-1)^n$ is bounded but not convergent.

Theorem 11.3.2 Order Limit Theorem

Let $(a_n)_{n\geqslant 1}$ be a sequence, assume $\lim_{n\to\infty} a_n = a$.

Moreover, suppose that $a_n \geqslant 0 \forall n \in \mathbb{N}$, then the conclusion is that $a \geqslant 0$. (That the limit is non negative too.)

Question 16

How would you try and convince me of this geometrically?

Solution: We will go by contradiction. Draw the line, and all my sequence is greater than 0. Assume then that the limit is strictly negative. But my sequence potentially gets very close to the limit. This implies that it has to cross the origin. Which is a contradiction.

Proof: Suppose for contradiction that $a < 0$.
We know: $\lim_{n\to\infty} a_n = a$, which means that:
$\forall \epsilon > 0 \exists N \in \mathbb{N}$ such that if $n \geqslant N$ then $|a_n - a| < \epsilon$.
We need to pick an ϵ that makes the whole thing go below zero.
Take the $|a| = \epsilon$, if we assume that a is negative, then $|a| > 0$.
In particular: for $\epsilon = -a$ which is positive because $a < 0$, there exists $N \in \mathbb{N}$ such that if $n \geqslant N$ then $|a_n - a| < -a$.
If we open this absolute value we get a contradiction.
This implies that $a < a_n - a < -a$. Consider only:
$a_n - a < -a$ for $n \geqslant N$. This gives us that $a_n < 0$ for $n \geqslant N$. This contradicts our assumption that the entire sequence is non negative.

😄

Theorem 11.3.3 Variations for the Order Limit Theorem

The thm. above is part 1 below is 2 and greater.

1. Suppose $(a_n), (b_n)$ two sequences such that $\lim_{n\to\infty} a_n = a, \lim_{n\to\infty} b_n = b$ and suppose that $a_n \geqslant b_n \forall n \in \mathbb{N}$.
   Then the conclusion is that $a \geqslant b$.

2. Suppose $\exists c \in \mathbb{R}$ such that $a_n \geqslant c \forall n \in \mathbb{N}$ then $a \geqslant c$.

Note:-

For one above consider the sequence $(a_n) - (b_n)$

Proof: Consider $c_n = a_n - b_n$. Then we know $c_n \geqslant 0$ for all $n \in \mathbb{N}$. Also (perhaps?) the $\lim_{n\to\infty} c_n = a - b$.

😄

Note:-

We will prove this next time.

Chapter 12

Thursday September 26th

Note:-
25 points of the midterm will be similar to the quiz we took on Wednesday September 25th

Note:-
Extra O.H. on Monday 1:00-2:30

Note:-
Preview Questions are on Canvas

12.1 Proving things about limits of sequences

Definition 12.1.1: Notation

If $\lim_{n\to\infty} a_n = a$, we can write $a_n \to a$.

Note:-
$a_n \to a$ means:
$\forall \epsilon > 0 \exists N \in \mathbb{N}$ such that if $n \geqslant N$, then $|a_n - a| < \epsilon$

Theorem 12.1.1 From Last Time

IF a_n converges to some limit then a_n is bounded.

Definition 12.1.2: Triangle Inequality

$|a - b| \leqslant |a| + |b|$

Definition 12.1.3: Inverse Triangle Inequality

$||a| - |b| \leqslant |a - b| \leqslant |a| + |b|$

Theorem 12.1.2

Let $(a_n), (b_n)$ be sequences such that $a_n \to a$ and $b_n \to b$. All of the following statements are true.

1. $a_n + b_n \to a + b$
2. $ca_n \to ca$ for all constants $c \in \mathbb{R}$

3. $a_nb_n \longrightarrow ab$

4. $\frac{a_n}{b_n} \longrightarrow \frac{a}{b}$ as long as $b \neq 0$.

Note:-

Two is very easy, because it follows from 3. Take b_n to be the constant sequence. Thus there is no real need to prove 2.

Note:-

Idea: For all of them, there are 2 limits that we knowand this means we can manipulate the $\epsilon > 0$. There is a limit that we want to show (We cannot limit the $\epsilon > 0$) in this case.

Proof: We will prove the first statement. Here we know that $a_n \longrightarrow a$ and $b_n \longrightarrow b$.
We want to show that $a_n + b_n \longrightarrow a + b$. Rewritten we want to show that $\forall\epsilon > 0\exists N$ such that if $n \geqslant N$ then $|a_n + b_n - a - b| < \epsilon$.
Let $\epsilon > 0$, (Start in this manner because we want to show for all; arbitrary but fixed).

Note:-

Scratch Work

$$|a_b + b_n - a - b| < \epsilon$$
$$|a_n - a + b_n - b|$$
$$|a_n - a + b_n - b| \leqslant |a_n - a| + |b_n - b|$$

because we know that $a_n \longrightarrow a$, we can make $|a_n - a|$ as small as I want. Similarly for $|b_n - b|$. Make each of them less than $\frac{\epsilon}{2}$ and then when added we know they are less than ϵ

Note:-

Formal Proof

Let $\epsilon > 0$. Since we know that $a_n \longrightarrow a$, $\exists N_1 \in \mathbb{N}$ such that if $n \geqslant N_1$, then $|a_n - a| < \frac{\epsilon}{2}$. We can do this because we can manpulate epsilon, we can do this for any epsilon of our liking by definition. Similarly for b_n.
Since we know $b_n \longrightarrow b$, $\exists N_2 \in \mathbb{N}$ such that if $n \geqslant N_2$ then $|b_n - b| < \frac{\epsilon}{2}$.
Set $N = \max\{N_1, N_2\}$, then if $n \geqslant N$, both 1 and 2 above are true.
Thus if $n \geqslant N$ we have:
$$|a_n + b_n - a - b| \leqslant |a_n - a| + |b_n - b|$$
Which gives us $< \frac{\epsilon}{2} + \frac{\epsilon}{2}$.

😊

Question 17

Explain the $N = \max\{N_1, N_2\}$

Solution: We need something that will be at least equal to the maximum for both of our definitions of $\frac{\epsilon}{2}$, for them to both be true simultaneously.

Note:-

We are proving the limit of the sum is the sum of the limit.

Proof: Proof of 3. We know that $a_n \longrightarrow a$ and $b_n \longrightarrow b$. We want to show that $a_nb_n \longrightarrow ab$.
This means we want to show: $\forall\epsilon > 0\exists N \in \mathbb{N}$ such that if $n \geqslant N$, then $|a_nb_n - ab| < \epsilon$.

Note:-

Scratch

Let $\epsilon > 0$. $|a_nb_n - ab|$, but we can not use the triangle inequality. Here we can add and subtract something so that we can create 4 terms and proceed as we did before.

$$|a_nb_n - ab| = |a_nb_n - a_nb + a_nb - ab|$$
$$|(a_nb_n - a_nb) + (a_nb - ab)|$$

Now apply triangle inequality:
$|(a_nb_n - a_nb) + (a_nb - ab)| \leqslant |a_nb_n - a_nb| + |a_nb - ab|$
$|a_n||b_nb| + |b||a_n - a|$
As before we can make $|a_n - a|$ as small as we want, and similarly $|b_n - b|$ we can make small.
We can set $|a_n - a| < \frac{\epsilon}{2|b|+1} = \epsilon_1$. We add the plus one because b could be zero. Adding one makes the guy even smaller so I am good.
The following is not a good choice because it depends on n: $\epsilon_2 = \frac{\epsilon}{2|a_n|+1}$.

Note:-

Remark: See Review Questions:
A sequence a_n is bounded if and only if $\exists M > 0$ such that $|a_n| \leqslant M \forall n \in \mathbb{N}$.

Because a_n is bounded below, set $\epsilon_2 = \frac{\epsilon}{2M}$.
Considering $|b_n - b|$, we know that

Note:-

Formal

Let $\epsilon > 0$. We know that $a_n \longrightarrow a$, thus a_n is bounded. This implies that $\exists M > 0$ such that $|a_n| \leqslant M$, $\forall n \in \mathbb{N}$. This is our first relation.
We know $a_n \longrightarrow a$, thus for $\epsilon_1 = \frac{\epsilon}{2|b|+1} > 0$, $\exists N_1 \in \mathbb{N}$ such that if $n \geqslant N_1$, then $|a_n - a| < \frac{\epsilon}{2|b|+1}$. This is our second relation.
We know $b_n \longrightarrow b$. Thus for $\epsilon_2 = \frac{\epsilon}{2M} > 0$, there exists $N_2 \in \mathbb{N}$ such that if $n \geqslant N_2$, then $|b_n - b| < \frac{\epsilon}{2M}$.
Set $N = \max\{N_1, N_2\}$. So for $n \geqslant N$, statements 1,2,3 above are true. Then by our scratch:
$|a_nb_n - ab| \leqslant |a_n||b_n - n| + |b||a_n - a| \leqslant$
$M|b_n - b| + |b||a_n - a|$ by relation 1
$\leqslant M\frac{\epsilon}{2M} + |b|\frac{\epsilon}{2|b|+1} \leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2}$.

☺

Proof: Proving 4. It is enough to prove if $b_n \longrightarrow b$, $b \neq 0$ then $\frac{1}{b_n} \longrightarrow \frac{1}{b}$. This is because we can use 3 above: $\frac{a_n}{b_n} = a_n\frac{1}{b_n} \longrightarrow a\frac{1}{b}$.
We know that $b_n \longrightarrow b$ and $b \neq 0$. We want to show that $\frac{1}{b_n} \longrightarrow \frac{1}{b}$. Meaning, $\forall \epsilon > 0 \exists N \in \mathbb{N}$ such that if $n \geqslant N$ then $|\frac{1}{b_n} - \frac{1}{b}| < \epsilon$.

Note:-

Scratch

Let $\epsilon > 0$.
$|\frac{1}{b_n} - \frac{1}{b}| = |\frac{b-b_n}{bb_n}|$
$\frac{|b-b_n|}{|b|b_n|} = \frac{|b_n-b|}{|b||b_n|}$
Here we know what to do with the top half. But for the bottom setting $|b_n| \leqslant M$ doesn't help.
This gives us $\frac{1}{|b_n|} \geqslant \frac{1}{M}$. This is completely useless.
What we want: To bound $|b_n|$ from below. By something positive. Meaning we want $|b_n| \geqslant m > 0$. So that when we flip them we get the right order. Because this implies that $\frac{1}{|b_n|} \leqslant \frac{1}{m}$.
Now we will apply the inverse triangle inequality. $||a| - |b| \leqslant |a - b| \leqslant |a| + |b|$
We know $b_n \longrightarrow b$, $b \neq 0$. Thus for $\epsilon_1 = \frac{|b|}{2} > 0$. We know this because $b \neq 0$. Then $\exists N_1 \in \mathbb{N}$ such that if $n \geqslant N$ then $|b_n - b| < \frac{|b|}{2}$. And in fact by the inverse triangle inequality; $||b_n| - |b|| \leqslant |b_n - b| < \frac{|b|}{2}$. Which implies that for $n \geqslant N_1$, $||b_n| - |b|| < \frac{|b|}{2} \longrightarrow -\frac{|b|}{2} < |b_n - |b| < \frac{|b|}{2}$.
My first relation is thus: $|b_n| > \frac{|b|}{2}$ for all $n \geqslant N_1$.

Note:-

Scratch

$|\frac{1}{b_n} - \frac{1}{b}| = \frac{|b_n-b|}{|b||b_n|} < |b_n - b|\frac{2}{|b|^2}$

Note:-

Formal

Let $\epsilon > 0$ since $b_n \longrightarrow b$, for $\epsilon_2 = \frac{\epsilon|b|^2}{2} > 0$, $\exists N_2 \in \mathbb{N}$ such that if $n \geqslant N_2$, then $|b_n - b| < \frac{\epsilon|b|^2}{2}$. As before set $N = \max\{N_1, N_2\}$. Then for $n \geqslant N$, both statements 1, 2 are true. This implies that $|\frac{1}{b_n} - \frac{1}{b}| = \frac{|b_n - b|}{|b||b_n|}$.

☺

Theorem 12.1.3 Full Version of the Order Limit Theorem

Assume $a_n \longrightarrow a$ and $b_n \longrightarrow b$.

1. If $a_n \geqslant 0$, $\forall n \in \mathbb{N}$ then $a \geqslant 0$.
2. If $a_n \geqslant b_n$, $\forall n \in \mathbb{N}$ then $a \geqslant b$
3. If $a_n \geqslant c$, $\forall n \geqslant 1$ where $c \in \mathbb{R}$ is some constant, then $a \geqslant c$

Proof: Proof for 2. Consider a new sequence, $c_n = a_n - b_n$. For this we know 2 things:

1. First $cn \geqslant 0$, because $a_n \geqslant b_n$ for all $n \in \mathbb{N}$
2. Since $a_n \longrightarrow a$ and $b_n \longrightarrow b$, the previous theorem tells us that $c_n \longrightarrow a - b$.

We can conclude here. By the first part we get that $a - b \geqslant 0 \longrightarrow a \geqslant b$.

☺

Note:-

Three is similar.

Theorem 12.1.4

If a_n converges, then a_n is bounded. We discussed that the converse is not true. Counter example being $a_n = (-1)^n$.

Can we make the converse true by adding something? The reason this guy went wrong was the osscilation, and the fact that there was a big gap.

Theorem 12.1.5 Monotone Convergence Theorem (MCT)

Let (a_n) be a sequence which is bounded and monotone. Then a_n converges.

Definition 12.1.4: Monotone

A sequence (a_n) is called increasing if $a_n \leqslant a_{n+1} \forall n \in \mathbb{N}$ and decreasing if $a_n \geqslant a_{n+1} \forall n \in \mathbb{N}$.

A sequence is monotone if it is one of the 2.

Chapter 13

Tuesday October 1st

Note:-

Midterm Thursday Office Hours on Wednesday from 3:30-5:00

Definition 13.0.1: Limit

$\lim_{n\to\infty} a_n = l$ if $\forall \epsilon > 0 \exists N \in \mathbb{N}$ such that if $n \geqslant N$ then $|a_n - l| < \epsilon$

Theorem 13.0.1 Monotone Convergence Theorem

Let (a_n) be a sequence which is bounded and monotone (meaning a_n is either increasing or decreasing).

Then the conclusion is that (a_n) converges.

Note that increasing means $a_n \leqslant a_{n+1}, \forall n \in \mathbb{N}$.

Question 18

Everything on the midterm about limits?

Solution: Everything that appears on the review. Prove something has a limit x or use the fact that x is the limit to show something else.

Note:-

Order limit theorem is the last thing on the midterm.

Look at the review everything that is there you should expect on the test.

Note:-

Last time you guessed that if a sequence is increasing and bounded above it will converge to the supremum.

Example 13.0.1

We will show that if (a_n) is increasing, it converges to its supremum.

Note:-

Do for practice, if a_n decreasing then it converges to its infimum.

Proof: Suppose (a_n) is increasing, meaning $a_n \leqslant a_{n+1} \forall n \in \mathbb{N}$, and (a_n) is bounded above.

Note:-

(a_n) increasing automatically implies that (a_n) is bounded below, that a_1 is a lower bound.

Let s =sup$\{a_n : n \in \mathbb{N}\}$, we will show that $a_n \to s$.
We want to show that $\forall \epsilon > 0 \exists N \in \mathbb{N}$ such that if $N \geqslant n$ then $|a_n| < \epsilon$.
Let $\epsilon > 0$, since s is the least upper bound $s - \epsilon$ is not an upper bound of $\{a_n : n \in \mathbb{N}\}$. This implies that $\exists N \in \mathbb{N}$ such that $a_n > s - \epsilon$.
For every $n \geqslant N$, we have the following
$a_N \leqslant a_n$ because sequence is increasing, we also have $s - \epsilon < a_N$, and we also have $a_n \leqslant s < s + \epsilon$. The last statement is true because s is an upper bound. Putting these statements toegether we have $\forall \geqslant N$ we have $s - \epsilon < a_n < s + \epsilon \to |a_n - s| < \epsilon$.

Note:-

Note that this time we used the ϵ neighborhood version of the definition.

😊

13.1 Applications of the MCT

Note:-

Powerful, because often we will get a sequence that is increasing and bounded above. But we don't have any way of computing the supremum. The MCT will tell us the sequence converges without finding, or solving for the supremum.

MCT allows us to show that some sequences converge without being able to compute the $\lim_{n\to\infty} a_n$.

Question 19

Is the converse true?
MCT says (a_n) monotone and bounded $\to$ convergence.

Solution: Consider something oscillating but still going to zero. Counterexample: $a_n = \frac{(-1)^n}{n}$. This converges to 0, therefore a_n is bounded, but it is clearly not monotone.

Example 13.1.1 (Application 1 Limits of Inductive Sequences)

Note:-

Will show one of the review problems

Let (a_n) be the sequence with $a_1 = 2, a_{n+1} = \sqrt{3 + a_n}$ for $n \geqslant 1$.

Note:-

We will prove by induction that this is bounded above and increasing. Then the MCT will tell us that the limit exists. Then we will use a trick to compute what the limit looks like.

CLAIM 1: (a_n) is increasing.

We will prove by induction on $n \in \mathbb{N}$ that $a_n \leqslant a_{n+1}$.

Proof: Base Case:
$a_1 = 2$
$a_2 = \sqrt{3+2} = \sqrt{5} > 2$.
Induction Hypothesis:
Suppose that $a_n \leqslant a_{n+1}$ for some $n \geqslant 1$.
Induction Step:
We will show $a_{n+1} \leqslant a_{n+2}$, start with the induction hypothesis and build it up.
$a_n \leqslant a_{n+1} \to a_n + 3 \leqslant a_{n+1} + 3$

$$\sqrt{a_n + 3} \leqslant \sqrt{a_{n+1} + 3}$$

😄

CLAIM 2: (Review Q)

Note:-

We want to show that the sequence is bounded but we need a statement that we can prove by induction. In order to show that the sequence is bounded, we need to provide some big upper bound.

Just pick a big number, say 100 or more. Say I will prove by induction that 100 is going to work.

Proof: We will prove by induction that $a_n \leqslant 100, \forall n \in \mathbb{N}$.
Base Case:
$a_1 = 2 < 100$
Induction Hypothesis:
Suppose $a_n \leqslant 100$ for some $n \in \mathbb{N}$
Induction Step:
We will show $a_{n+1} \leqslant 100$
$a_n \leqslant 100 \longrightarrow a_n + 3 \leqslant 103 =$
$\sqrt{a_n + 3} \leqslant \sqrt{103} < 100$
$\longrightarrow a_{n+1} < 100$
So we can conclude that 100 is an upper bound of $\{a_n : n \in \mathbb{N}\}$ and hence (a_n) is bounded above.

😄

Now by MCT, we have a_n converges to a limit: $l = \lim_{n\to\infty} a_n$.

Next: We want to compute the l.

Note:-

Now that we know it exists we can use a trick.

What we know:

We know $a_n \longrightarrow l$
The sequence $b_n = a_{n+1}$ also converges to l, (if you don't see why do it for practice for the midterm Prove this),
But $b_n = \sqrt{a_n + 3}$, therefore $b_n \longrightarrow \sqrt{l + 3}$
Thus $b_n \longrightarrow l$ and $b_n \longrightarrow \sqrt{l + 3}$.

Note:-

Last Review Problem (18) says that the limit of a sequence is unique.

This means that we can solve:
$l = \sqrt{l + 3} \longrightarrow l^2 - l - 3 = 0 \longrightarrow$ by the quadratic formula: $l = \frac{1 \pm \sqrt{13}}{2}$, then we can conclude that $a_{\rightarrow} \frac{1+\sqrt{13}}{2}$.

Note:-

We only take the positive sqrt for square root technicalities.

13.2 Infinite Series

Definition 13.2.1: Infinite Series

Let (a_n) be a sequence. We form a new sequence s_n, which is called the sequence of partial sums of a_n, defined as $s_n = a_1 + a_2 + ... + a_n$.
$s_n = \Sigma_{k=1}^{n} a_k$.

Definition 13.2.2: Infinite Series

The infinite series $\Sigma_{n=1}^{\infty} a_n$ is the $\lim_{n\to\infty} s_n$ if this limit exists.
In other words: We say that the infinite series $\Sigma_{n=1}^{\infty} = l$ or converges to l if $lim_{n\to\infty} s_n = l$.

If $\lim_{n\to\infty} s_n$ does not exist, we say that $\Sigma_{n=1}^{\infty} a_n$ diverges.

Example 13.2.1 (Application 2 Sums of Infinite Series (MCT))

Let (a_n) be a sequence such that $a_n \geqslant 0$, $\forall n \in \mathbb{N}$. Then $s_n = a_1 + ... + a_n$, the sequence of partial sums.

The sequence of partial sums is automatically increasing now.

Proof: We want to show that $s_n \leqslant s_{n+1} \forall n \in \mathbb{N}$
This is equivalent to showing $s_{n+1} - s_n \geqslant 0 \forall n \in \mathbb{N}$
$s_{n+1} = a_1 + ... + a_n + a_{n+1}$
$s_n = a_1 + ... + a_n$
Thus, $s_{n+1} - s_n = a_{n+1} \geqslant 0$.
If we start with a sequence of non negative numbers, the sequence of partial sums is always increasing. If we can find some upper bound then the sequence converges by the MCT.

☺

Example 13.2.2 (Example 1)

Consider $s_n = \Sigma_{n=1}^{\infty}(-1)^n$, this implies that $a_n = (-1)^n$. We form the sequence of partial sums $s_n = a_1 + ... + a_n$

$s_n = 0$ if n is even and -1 if n is odd.

s_n diverges, which implies that $\Sigma_{n=1}^{\infty}(-1)^n$ diverges.

Example 13.2.3 (Example 2 A telescoping Series)

We want to explore $\Sigma_{n=2}^{\infty} \frac{1}{n(n-1)}$. Hand wavey-way that we saw in high school, was to split this fraction up:

$\Sigma_{n=2}^{\infty} \frac{1}{n-1} - \Sigma_{n=2}^{\infty} \frac{1}{n} =$ where everything cancels out except the first term.

Note:-
Formal Way

Proof: First note that $\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$.
Form the sequence of partial sums:
$s_n = a_1 + a_2 + ... + a_n$
$= \Sigma_{k=1}^{\infty} \frac{1}{k(k-1)} = \Sigma_{k=2}^{n} (\frac{1}{k-1} - \frac{1}{k}) = \Sigma k = 2^n \frac{1}{k-1} - \Sigma_{k=2}^{n} \frac{1}{k}$

From here we see that everything cancels out except for the first and the last term.
$1 - \frac{1}{n}$. Now we know that $s_n = 1 - \frac{1}{n}$.
$\lim_{n\to\infty} s_n = \lim_{n\to\infty}(1 - \frac{1}{n}) = \lim_{n\to\infty} - \lim_{n\to\infty} \frac{1}{n} = 1$

😊

Example 13.2.4

$\frac{1}{n(n+3)}$ how do we split this up into two fractions.

Look for $A, B \in \mathbb{R}$ such that $\frac{1}{n(n+3)}$ such that it equals $\frac{A}{n} + \frac{B}{n+3}$.

$1 = A(n+3) + Bn$, pick two different values for n and solve the system of equations.

Example 13.2.5

Let (a_n) be a sequence give a definition of:
$\lim_{n\to\infty} a_n = \infty$, or in words a_n diverges to ∞

Chapter 14

Tuesday October 8th

Note:-

No quiz this week but there will be a quiz next week about convergence of series.

14.1 Reminders

Definition 14.1.1

a_n a sequence. We form a new sequence called a sequence of partial sums of (a_n) is $S_n = a_1 + ... + a_n$. In other words $S_n = \Sigma_{k=1}^{n} a_k$.

Definition 14.1.2

We define the infinite series to be $\Sigma_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$ if it exists we say that $\Sigma_{n=1}^{\infty} a_n$ diverges

Note:-

Last time we discussed that telescoping series converges

Note:-

And that $\Sigma(-1)^n$

Example 14.1.1 (The Harmonic Series)

$\Sigma_{n=1}^{\infty} \frac{1}{n}$

Note:-

Claim is that this diverges.

We want to show that $\Sigma_{n=1}^{\infty} \frac{1}{n} = \infty$, equivalently, we want to show that s_n diverges.

Note:-

Idea: This one needs a fancy trick. We will start with $s_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ now this is bigger than or equal to $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + 2\frac{1}{2}$

Now we will go to the next power of 2, $s_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$ then we will plug in $\frac{1}{8}$ and create an inequality. The above implies:

$s_8 \geqslant 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \longrightarrow S_{2^3} \geqslant 1 + 3\frac{1}{2}$.

Note:-

From this point we guess $\forall n \geqslant 1$, $s_{2^n} \geqslant 1 + n\frac{1}{2}$

If The above is true, then s_n will not be bounded above, and therefore s_n diverges.

Note:-

Begin Proof by induction

1. True for $n = 1$

2. Induction Hypothesis: Suppose that $s_{2^n} \geqslant 1 + \frac{n}{2}$

3. We want to show that $s_{2^{n+1}} \geqslant 1 + \frac{n+1}{2}$
   The LHS $= s_{2^{n+1}} = 1 + \frac{1}{2} + \ldots + \frac{1}{2^n} + \frac{1}{2^n+1} + \frac{1}{2^n+2} + \ldots + \frac{1}{2^{n+1}}$
   Because we know that all of the earlier terms are greater than $\frac{1}{2^{n+1}}$ we can then say that
   $s_{2^{n+1}} \geqslant 1 + \frac{n}{2} + \frac{2^n}{2^{n+1}} = 1 + \frac{n+1}{2}$

Question 20: Aside

We say that a sequence (a_n) diverges to ∞ and write $a_n \longrightarrow \infty$ or $\lim a_n = \infty$ if:

Solution: $\forall M > 0 \exists N \in \mathbb{N}$ such that if $n \geqslant N$ then $a_n > M$.

The claim is that this sequence gets arbitrarily large.

Question 21: Practice

State $\lim_{n\to\infty} a_n = -\infty$

Example 14.1.2 (Geometric Series)

$\Sigma_{n=0}^{\infty} r^n$

Question 22: What is going on with the sequence inside?

Solution: $r \in \mathbb{R}$ a fixed real number, the answer will depend on what r is.

Start with the sequence $a_n = r^n$. We want to describe when this sequence converges and when it doesn't.

Note:-

Known Values

1. $r = 0, a_n = 0 \longrightarrow a_n \longrightarrow 0$

2. $r = 1, a_n = 1 \longrightarrow a_n \longrightarrow 1$

3. $r = -1, a_n = (-1)^n \longrightarrow$ diverges

4. We can make a guess that in the open interval $(-1, 1)$ the sequence will converge to 0.

5. We guess that outside of this range elements will dieverge to ∞ for the positive values in the reals.

Note:-

First we will start with the cases that converge to 0

Take for granted what we know. Let $r \in (0, 1)$ fixed. We have the sequence $a_n = r^n$ and we want to show that $a_n \longrightarrow 0$.

1. $r > 0 \longrightarrow a_n > 0 \longrightarrow a_n$ is bounded below by 0

2. a_n is strictly decreasing Consider $a_{n+1} = r^{n+1}$ which is $r \cdot r^n = ra_n$, but now $0 < r < 1$ and a_n is positive $\longrightarrow 0 < ra_n < a_n$ by multiplying through. This implies that $a_{n+1} < a_n$.

By the monotone convergence theorem, a_n converges (and to its infimum). It remains to show that this limit is zero. Let $L = \lim_{n\to\infty} a_n$, we want to show that $l = 0$. We know that $a_n \longrightarrow l$ and that $a_{n+1} \longrightarrow l$. But $a_{n+1} = r \cdot a_n$ by algebra of limits we get that $a_{n+1} \longrightarrow r \cdot l$. Because the limit is unique we know that $l = r \cdot l \longrightarrow l = 0$ because $r \neq 1$.

Note:-

Let $r \in (-1, 0)$

Again we want to show that $r^n \longrightarrow 0$. This means that $\forall \epsilon > 0 \exists N \in \mathbb{N}$ such that if $n \geqslant N$, then $|r^n - 0| < \epsilon$.

But $|r^n - 0| = |r^n| = |r|^n$, since $|r| \in (0, 1)$ we know that $|r|^n \longrightarrow 0$ by the previous problem

Note:-

Let $r > 1$

Then $r > 1 \longrightarrow 0 < \frac{1}{r} < 1 \longrightarrow (\frac{1}{r})^n \longrightarrow 0$. If this is true then $r^n \longrightarrow \infty$.

$a_n \longrightarrow \infty$ if: $\forall M > 0 \exists N \in \mathbb{N}$ such that if $n \geqslant N$, then $a_n > M$.

$(\frac{1}{r})^\infty \longrightarrow 0$ goes to zero because $\frac{1}{r} \in (-1, 1)$. We want to show that $r^n \longrightarrow \infty$, meaning that r^n gets arbitrarily large as $n \longrightarrow \infty$. We know that $(\frac{1}{r})^n$ is equal to $\frac{1^n}{r^n}$ which means that r^n gets arbitrarily small as $n \longrightarrow \infty$.

Note:-

Let $r < -1$

Set $x = -r \longrightarrow x > 1$. $a_n = r^n = (-x)^n = (-1)^n x^n$, for n even, $a_n = x^n$ and x^n diverges to ∞, this is enough to say that a_n is not bounded above. Then a_n diverges becuase it is not bounded above.

Note:-

This was an exploration of r^n

Example 14.1.3 (Geometric Series)

$a_n = r^n, n \geqslant 0$, the sequence of partial sums. $s_n = r^0 + r^1 + ... + r^n$, equivalently $s_n = 1 + r + r^2 + ... + r^n$. $s_{n+1} = 1 + r + r^2 + ... + r^n + r^{n+1}$, it is always true that $s_{n+1} - s_n = r^{n+1}$

Note:-

rs_n is almost $sn + 1$.

$rs_n = r + r^2 + ... + r^{n+1} \longrightarrow r \cdot s_n = 1 + r + r^2 + ... + r^{n+1}$

This gives us $1 + r \cdot s_n = s_{n+1}$

This gives us $s_{n+1} = s_n + r^{n+1}$. Then using our earlier equation we have: $s_n + r^{n+1} = 1 + r \cdot s_n \longrightarrow s_n = \frac{1-r^{n+1}}{1-r}$, this is true if $r \neq 1$, $r = 1$ will be treated separately.

By algebra of limits, $\lim_{n\to\infty} s_n = \frac{1-r\lim r^n}{1-r}$

Note:-

The conclusion is

$\Sigma_{n=0}^{\infty} r^n =$

1. $\frac{1}{1-r}$ if $r \in (-1, 1)$
2. diverges if $r \geqslant -1$
3. diverges For $r > 1$

Note:-

$r = 1$, $s_n = 1 + 1 + 1 ... + 1$ and $s_n = n + 1$. This implies that $s_n \longrightarrow \infty$

Note:-

We have shown two infinite examples, and calcualated the limits for them, telescoping and harm

Theorem 14.1.1 Comparison Test

Let (a_n) and (b_n) be sequences. $0 \leqslant a_n \leqslant b_n$, $\forall n \geqslant 1$. Then if b_n converges, it will force the smaller tequence to convergence. If the smaller sequence divereges it will force the bigger one to do so as well.

Proof: Let $s_n = a_1 + a_2 + ... + a_n, t_n = b_1 + b_2 + ... + b_n$ be the sequences of partial sums of a_n, b_n respectively.

$a_n \leqslant b_n \forall n \in \mathbb{N} \longrightarrow s_n \leqslant t_n, \forall n \in \mathbb{N}$.

$a_n, b_n \geqslant 0$ implies that s_n, t_n are both increasing. Now we assume that $\lim_{n\to\infty} L$ exists.

t_n converges $\longrightarrow t_n$ is bounded. This implies that t_n converges to its supremum. This implies that $t_n \leqslant L \forall n \in \mathbb{N}$.

That was the last bit of information that we need. Because $s_n \leqslant t_n$ we know that by the first two pieces of information, s_n is bounded above, that is $0 \leqslant s_n \leqslant L \forall n \in \mathbb{N}$. Because$s_n$ is is increasing and bounded above, we can apply MCT. We get htat s_n converges and converges to its supremum.

😄

Note:-

Bi-product of the proof: that if $M = \lim_{n\to\infty} s_n = \Sigma_{n=1}^{\infty} a_n$ and $L = \lim_{n\to\infty} t_n = \Sigma_{n=1}^{\infty} b_n$ then $M \leqslant L$.

Chapter 15

Thursday October 10th

15.1 Reminders

1. Telescoping Series Converges
2. $\Sigma_{n=1}^{\infty} \frac{1}{n}$ diverges
3. Geometric series $\Sigma_{n=0}^{\infty} = \frac{1}{1-r}$, if $-1 < r < 1$.
4. And we proved the Comparison Test

Theorem 15.1.1 Comparison Test

Suppose $0 \leqslant a_n \leqslant b_n$, For all $n \in \mathbb{N}$

1. If Σb_n converges then Σa_n converges too.
2. If Σa_n diverges then Σb_n diverges

Example 15.1.1

$\Sigma_{n=1}^{\infty} \frac{1}{n^2}$, want to show that this converges. We want to find something larger and use the comparison Test.

Proof: For every $n \geqslant 2$, we have: $u \leqslant \frac{1}{n^2} \leqslant \frac{1}{n(n-1)}$. By the Comparison Test we get that $\Sigma_{n=2}^{\infty} \frac{1}{n^2}$ converges.

But we can add the first term and get what we wanted. $\Sigma_{n=1}^{\infty} \frac{1}{n^2} = 1 + \Sigma_{n=2} \frac{1}{n^2}$.

😊

Example 15.1.2

$\Sigma_{n=1}^{\infty} \frac{1}{n!}$

We know that $\frac{1}{n!} \frac{1}{1 \cdot 2 \cdot \ldots \cdot n} \leqslant \frac{1}{1 \cdot 2 \cdot \ldots \cdot 2}$. $\Sigma_{n=1}^{\infty} = \frac{1}{2^{n-2}}$ converges because its geometric. By comparison $\Sigma_{n=1}^{\infty} \frac{1}{n!}$ converges.

Note:-

The first few terms don't matter in regards to whether or not an infinite series converges.

Note:-

Instead they matter if we want to compute the actual sum.

Note:-

As a result, if we only care about convergence, I will write Σa_n without specifying where I start.

Question 23

$\Sigma_{n=1}^{\infty} a_n$ converges $\leftrightarrow$ $\Sigma_{n=k}^{\infty} a_n$ converges.

Theorem 15.1.2 Algebraic Limit Theorem For infinite Series

Suppose we have $\Sigma_{n=1}^{\infty} a_n, \Sigma_{n=1}^{\infty} b_n$, both converge. Additionally suppose that $\Sigma_{n=1}^{\infty} a_n = A$ and that $\Sigma_{n=1}^{\infty} b_n = B$. Then $\Sigma_{n=1}^{\infty}(ca_n + db_n)$ converges to $ca + dB$ for $c, d \in \mathbb{R}$.

This gives us $\Sigma_{n=1}^{\infty}(ca_n + db_n) = c\Sigma_{n=1}^{\infty} a_n + d\Sigma_{n=1}^{\infty} b_n$

Proof: Consider the sequences of partial sums $s_n = a_1 + ... + a_n$ and $t_n = b_1 + ... + b_n$. For a_n, b_n. By assumption we have $S_n \longrightarrow A$, $t_n \longrightarrow B$.By the Algebraic Limit Theorem for sequences we get that $c \cdot s_n + d \cdot t_n \longrightarrow cA + dB$. Because $c \cdot s_n + d \cdot t_n$ is the sequence of partial sums for $ca_n + db_n$ we can conclude. 😊

Note:-

15 points on the next midterm, determine which of the following series converge or diverge.

Definition 15.1.1: Divergence Test

Let a_n be a sequence if $\Sigma_{n=1}^{\infty} a_n$ converges then $a_n \longrightarrow 0$.

Note:-

This test is helpful to show certain series diverge

Theorem 15.1.3 Contrapositive of Divergence Test

If a_n does not converge to 0. Then Σa_n diverges. Example, $\Sigma(-1)^n$ diverges because the guy inside does not go to zero.

Or $\Sigma 3^n$ diverges because 3^n does not go to zero.

Proof: $\Sigma_{n=1}^{\infty} a_n$ converges which means that the sequence of partial sums $s_n = a_1 + ... + a_n$ converges to L. By definition. $S_n \longrightarrow L$ implies that $S_{n+1} \longrightarrow L$. Equivalently $S_{n+1} = S_n + a_{n+1} = a_{n+1} = S_{n+1} - S$. This gives us $S_n \longrightarrow L$ $S_{n+1} \longrightarrow L$ this gives us $a_{n+1 \to L-L=0}$. Applying the Algebraic Limit Theorem implies that $a_n \longrightarrow 0$. 😊

Question 24

Is the converse True?

Solution: No; Consider the Harmonic Series.

Note:-

Many students want to use the converse. Be careful not to use this argument. The sequence inside goes to zero doesn't give us any information.

Note:-

The first check you should do is this test. Sometimes we forget about this test, this is why it is sneaky.

Definition 15.1.2: Absolute Convergence Test

If The series of $\Sigma|a_n|$ converges, then Σa_n converges.

Theorem 15.1.4

If $\Sigma_{n=1}^{\infty}|a_n|$ converges $\longrightarrow \Sigma a_n$ converges.

In this case we say that $\Sigma_{n=1}^{\infty} a_n$ converges absolutely.

Proof: Know $\Sigma|a_n|$ converges.

Note:-

WRONG PROOF

$a_n \leqslant |a_n|, \forall n$, By the comparison Test Σa_n converges

Note:-

This is wrong because we don't know that $a_n > 0$. We can't apply comparison test, it will be false otherwise.

$-|a_n| \leqslant a_n \leqslant |a_n| \forall n \in \mathbb{N}$. To make everything go to the positive add $|a_n|$: $0 \leqslant a_n + |a_n| \leqslant 2|a_n|$. Now we can apply the comparison test starting with the values greater than 0. Thus, $\Sigma(a_n + |a_n|)$ converges because $\Sigma 2|a_n| = 2\Sigma|a_n|$ converges.

$\Sigma(a_n + |a_n|)$ converges, $\Sigma|a_n|$ converges, and so by the Algebraic Limit theorem $\Sigma - |a_n|$ converges and $\Sigma(a_n + |a_n| - |a_n|$ converges to Σa_n.

😊

Note:-

Theorem 15.1.5 Ratio Test

Will be on the next homework.

Example 15.1.3

$\Sigma_{n=1}^{\infty} \frac{1}{n^p}$ where $p > 0$ is fixed.

$\Sigma \frac{1}{n}$ diverges which implies that $\Sigma \frac{1}{n^p}$ diverges for $\forall p \leqslant 1$.

Also, $\Sigma \frac{1}{n^2}$ converges which implies that $\Sigma \frac{1}{n^p}$ converges $\forall p \geqslant 2$. By the Comparison Test

Question 25

What about $p(1,2)$. We need: Cauchy Condensation Test.

Definition 15.1.3: Cauchy Condensation Test

Suppose (b_n) is a decreasing sequence. Such that $b_n \geqslant 0$ for all $n \geqslant 1$. Then $\Sigma_{n=1}^{\infty} b_n$ converges if and only if $\Sigma 2^n b_2 n$ converges

Example 15.1.4 (Application)

$\Sigma \frac{1}{n^p}$ converges $\forall p > 1$. Follow weird criterion.

Proof: $\Sigma \frac{1}{n^p}$, $b_n = \frac{1}{n^p} > 0$, decreasing.

$2^n \cdot b_2 n = 2^n \cdot \frac{1}{2^{np}} = \frac{1}{2^{n(p-1)}}$

Determine whether $\Sigma \frac{1}{2^{n(p-1)}}$ converges. ***Solution:*** Consider $r = \frac{1}{2}$ $n = -n(p-1)$ $r = \frac{1}{2}$ so this series converges.

$\Sigma(\frac{1}{2^{p-1}})^n$ set $r = \frac{1}{2^{p-1}}$ So this becomes a geometric series. Then this converges if and only if $\frac{1}{2^{p-1}} \in (-1, 1)$ or if $p > 1$

😃

Note:-

The harmonic is the first infinite series that diverges

Definition 15.1.4: Ratio Test

Note:-

Good Practice for next week's quiz

Example 15.1.5

Determine which of the following series converge.

1. $\Sigma_{n=3}^{\infty} \frac{11}{2^{n+1}}$Make it look like geometric. Believe it should converge. $= 11 \cdot \Sigma_{n=3} \frac{1}{2 \cdot 2^n} = \frac{11}{2} \Sigma_{n=3}^{\infty} \frac{1}{2^n}$. Which we know converges. To compute the infinite sum, we know that $\Sigma_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = 2$. Then: $\Sigma_{n=3}^{\infty} \frac{11}{2^{n+1}} = \frac{11}{2} \Sigma_{n=3}^{\infty} \frac{1}{2^n} = \frac{11}{2} (\Sigma_{n=0}^{\infty} \frac{1}{2^n} - 1 - \frac{1}{2} - \frac{1}{2^2})$...
2. $\Sigma_{n=1}^{\infty} ((\frac{-3}{4})^n + \frac{5}{n})$ Guess that this diverges. There are two parts and the left part converges and the right part diverges. We want to show the series diverges. Suppose for contradiction that it converges. Then we have $\Sigma((\frac{-3}{4})^n + \frac{5}{n})$ converges and note that $\Sigma(\frac{-3}{4})^n$ converges as geometric. By the ALT we should be able to subtract the too. Which gives us that $\Sigma \frac{5}{n}$ converges implying that $\Sigma \frac{1}{n}$ converges. A Contradiction!
3. $\Sigma_{n=1}^{\infty} \frac{n}{n+1}$ Diverges because the sequence inside converges to 1 and not 0. Thus divergence test.
4. $\Sigma_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ $|a_n| = \frac{1}{n^2}$. $a_n = \frac{(-1)^{n+1}}{n^2}$. We know that $\Sigma \frac{1}{n^2}$ converges, which means that the $\Sigma_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges absolutely. Use Absolute convergence Test.
5. $\Sigma \frac{(-1)^n}{n}$, we can immediately use the alternating series test. Thus $\Sigma \frac{(-1)^n}{n}$ converges.

   Note:-

   Notice $\Sigma \frac{(-1)^n}{n}$ converges, BUT, $\Sigma | \frac{(-1)^n}{n} | = \Sigma \frac{1}{n}$ diverges. Then the converse of the absolute convergence test is not true

Definition 15.1.5: Alternating Series Test

Suppose (a_n) is a sequence satisfying the following:

1. $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n$
2. $a_n \longrightarrow 0$ goes to zero

Then the alternating series $\Sigma(-1)^n a_n$ converges

15.2 Tests

1. Telescoping Series Converge
2. $\Sigma_{n=1}^{\infty} \frac{1}{n}$ diverges
3. Geometric Series $\Sigma_{n=0}^{\infty} r^n = \frac{1}{1-r}$, if $-1 < r < 1$.
4. Comparison Test
5. Divergence Test

6. Absolute Convergence Test
7. Ration Test
8. $\Sigma \frac{1}{n^p}$ converges if and only if $p > 1$
9. Alternating Series Test

Definition 15.2.1

If $\Sigma|a_n|$ converges we say Σa_n converges absolutely. If Σa_n converges but $\Sigma|a_n|$ diverges we say that Σa_n converges conditionally.

Note:-

$a_n \longrightarrow a$ if $\forall \epsilon > 0 \exists N \in \mathbb{M}$ such that if $n \geqslant N$ then $|a_n - a| < \epsilon$.

Definition 15.2.2

Let (a_n) be a sequence. We say that a_n is cauchy if $\forall \epsilon > 0 \exists N \in \mathbb{N}$ such that for all $n, m \geqslant N$, $|a_n - a_m| < \epsilon$.

Note:-

Maybe we don't know a limit but we know that after some point the terms get very close to each other.

Theorem 15.2.1

What we will show is that $a_n \longrightarrow a$ if and only if a_n is cauchy.

Note:-

Cauchy is the person who came up with all of this notation.

Chapter 16

Thursday October 17th

Note:-

Finish Chapter 2 Today

Note:-

Pace will go on faster

Note:-

Chapters 3,4,5 will be similar to what we've seen

16.1 Highlights so far

Note:-

1. Archimedean Property
2. Density of Rationals
3. Nested Interval Property
4. Monotone Convergence Theorem

Today we will add the Bolzano-Weierstrass Theorem, which is of equal importance

Note:-

3,4 and the BW theorem are sufficient to replace the axiom of completeness

Definition 16.1.1: Subsequences

Let (a_n) be a sequence. Suppose that we have a strictly increasing sequence of natural numbers: $k_1 < k_2 < k_3 < ... < k_n$ and so on, and we think of these as indices.

We can form a new sequence $b_n = a_{k_n}$, meaning

$$b_1 = a_{k_1}$$
$$b_2 = a_{k_2}$$

We call these b_i subsequences of a_n

16.2 Examples

Example 16.2.1 (Example 1)
Let $a_n = \frac{1}{n}$

$$\{1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}\}$$

Question 26

Determine which of the following are subsequences of a_n

1. $\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{6}, \frac{1}{2}, \frac{1}{10}, ...\}$
2. $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, ...\}$
3. $\{\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, ...\}$
4. $\{\frac{1}{p} :$ p is prime$\}$
5. $\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}\}$

The first one is not a subsequence because the indices are not strictly increasing.
The second is a subsequence consider $b_n = a_{2n}$
The third one is also a subsequence consider $b_n = a_{10^n}$
Yes for primes but we can't find an explicit formula for k_n
No indexes are not strictly increasing. Not strictly increasing implies not a subsequence. Consider.

Theorem 16.2.1
Let (a_n) be a sequence. Suppose $a_n \to a$. Then every subsequence of (a_n) also converges to a.

Proof: Let (a_{k_n}) be a subsequence of a_n, we want to show that $a_{k_n} \to a$. We want to show that $\forall \epsilon > 0 \exists N \in \mathbb{N}$ such that if $n \geqslant N$ then $|b_n - a| < \epsilon$
Let $\epsilon > 0$. Becuase we know $a_n \to a$, $\exists N \in \mathbb{N}$ such that if $n \geqslant N$ then $|a_n - a| < \epsilon$. But by the lemma below, we get that $k_n \geqslant n$. So if $n \geqslant N$ then $k_n \geqslant N \to |a_{k_n} - a| < \epsilon \to |b_n - a| < \epsilon$.

😊

Note:-
Subsequences are infinite. The only rule is that the indices are strictly increasing.

Note:-
Make sure you can write this lemma down. This lemma is very useful with the contrapositive.

Note:-
If not all subsequences of a_n converge to a, then a_n does not converge to a either

Example 16.2.2 (What Isaac said as a particular example)
If a_n has two subsequences a_{k_n} and a_{l_n} that converge to different limits, then a_n doesn't converge.

Example 16.2.3
If a_n has a subsequence that diverges, take for example diverges to infinity, then a_n diverges itself.

Example 16.2.4

Take $a_n = 5-(-1)^n$. Immediately if we take the even terms, $a_{2n} = 5-1 = 4$, but if we take the odd terms, $a_{2n+1} = 5+1 = 6$, and since these two subsequences converge to different values, we know that a_n diverges.

Note:-

If we want to show that a sequence diverges, all we want to do is produce two subsequences that diverge to $\pm\infty$ or like the ones above.

Note:-

Lemma's are either easier or prepatory results to make theorems easier.

Lenma 16.2.1

Let $b_n = a_{kn}$ be a subsequence of a_n. Then the claim is that the index $k_n \geqslant n, \forall n \in \mathbb{N}$

Proof: By induction on $n \in \mathbb{N}$

Base Case:
We want to show that $k_1 \geqslant 1$. We are indexing from the natural numbers. This is clear since k_1 is a natural number.
Induction Hypothesis:
Supoose $k_n \geqslant n$ for some $n \geqslant 1$
Induction Step:
We want to show that $k_{n+1} \geqslant n+1$. By the induction hypothesis we have $k_n \geqslant n$. Also we have that $k_{n+1} > k_n$ because a_{k_n} is a subsequence. By transitivity we have the following:
$k_{n+1} > n$ this implies that $k_{n+1} > n+1$. This is sufficient because if k_{n+1} is in the naturals and is strictly greater than n, it must be the case that it is greater than or equal to $n+1$, which is also a natural number.

😊

16.3 Big Theorem (BW)

Theorem 16.3.1 Bolzano-Weierstrass Theorem

Every Bounded Sequence, (a_n), has a convergent subsequence. Meaning, if (a_n) is bounded, then $\exists(a_{k_n})$ subsequence that has a limit.

16.3.1 Remarks

Note:-

We know that Convergent implies bounded. We know that the converse is not true. Take $(-1)^n$

Note:-

Notice that $(-1)^n$ does have subsequences that converge to something.

Note:-

Think that BW gives a "Partial Converse". If it is bounded we can not say that it converges but we can say that it has subsequences that converge.

16.3.2 Preparation

Definition 16.3.1: Nested Interval Property

Let $I_n = [a_n, b_n]$, $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n$ be a nested sequence of closed intervals. Then $\cap_{n=1}^{\infty} I_n \neq \emptyset$

16.3.3 True False

1. The Nested Interval Property implies that $\cap_{n=1}^{\infty} = \{x\}$
   Counter Example: $[0, 1+\frac{1}{n}]$, then $\cap_{n=1}^{\infty} = [0,1]$.

2. Can we add an assumption to make the statement true?
   Yes. See Theorem Below (Proven in HW7)

Theorem 16.3.2

Let $I_n = [a_n, b_n]$ be a nested sequence of closed intervals, and suppose that the length of $I_n = b_n - a_n \longrightarrow 0$. Then we can conclude that $\cap_{n=1}^{\infty} I_n = \{x\}$.

16.3.4 Proof of BW Theorem

Note:-

Saw a proof less technical to this when we proved the set of real numbers is uncountable.

Proof: Given. Sequence (a_n) which is bounded $\rightarrow \exists M > 0$ such that $|a_n| \leqslant M, \forall n \in \mathbb{N} \rightarrow -M \leqslant a_n \leqslant M, \forall n \in \mathbb{N}$.

Note:-

Our strategy is to construct a nested sequence of closed intervals $I_n = [a_n, b_n]$ with $b_n - a_n \longrightarrow 0$ and a subsequence a_{k_n} with $a_{k_n} \in I_n, \forall n$.

Set $I_0 = [-M, M]$. The length of $I_0 = 2M$ and $a_n \in I_0 \forall n \in \mathbb{N}$.

1. STEP 1: Split I_0 into two halves. The midpoint of course is zero. We can split it into two intervals, $[-M, 0][0, M]$. At least one of the two halves must contain infinitely many terms of the sequence. Pick the half that contains infinitely many a_n's and we will call this I_1. Set $I_1 = [a_1, b_1]$ which means $[-M, 0]$ or $[0, M]$. I_1 contains infinitely many a_n's. Chose 1 term of a_n in I_1 and write it a_{k_1}.

   Note:-

   The length of $I_1 = \frac{1}{2} I_0 = M$

2. So far we have constructed $I_1 \supseteq I_0$ with $b_1 - a_1 = \frac{\text{length}(I_0)}{2}$ and $a_{k_1} \in I_1$.

   Note:-

   We will call the sequence x_n to avoid confusion with the endpoints of the intervals to avoid confusion from this point forward.

3. STEP 2: Split I_1 into 2 halves. At least one of the 2 will contain infinitely many x_n's. Pick one that does and call it $I_2 = [a_2, b_2]$. Now we need to pick a_{k_2}. The claim is that we can pick $x_{k_2} \in I_2$ with $k_2 > k_1$.

   Note:-

   Contradiction why not

   Suppose not. If $x_n \in I_2$, then $n \leqslant k_1 \longrightarrow I_2$ contains only finitely many terms which is a contradiction.

   Note:-

   Return to step 2

   So far we have $I_0 \supseteq I_1 \supseteq I_2$ with length of $I_2 = \frac{1}{2}$ length$(I_1) = \frac{1}{4}$ length(I_0), and we have picked $x_{k_1} \in I_1$, $x_{k_2} \in I_2$ with $k_2 > k_1$.

4. Continue Inductively: Suppose that for some $n \geqslant 1$, we have constructed closed intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n$, where $I_i = [a_i, b_i]$, $1 \leqslant i \leqslant n$, such that each I_i contains ∞ many x_n's. The length of $I_n = \frac{1}{2^n}$ length(I_0), and we have picked terms $x_{k_i} \in I_i$ with indices $k_1 < k_2 < \ldots < k_n$, then we construct the next interval I_{n+1} as follows: We split I_n into two halves, pick a half that contains ∞ many terms and we call it I_{n+1} This

implies that the length of $I_{n+1} = \frac{1}{2}$ length (I_n). This iplies that the length $(I_{n+1}) = \frac{1}{2^{n+1}}\text{length}(I_0)$. Since I_{n+1} contains ∞ many x_n's, we can find $x_{k_{n+1} \in I_{n+1}}$ such that $k_{n+1} > k_n$.

5. This way we constructed a nested sequence of closed intervals $I_0 \supseteq I_1 \supseteq \ldots \supseteq I_n$ with $I_n = [a_n, b_n]$ with $b_n - a_n = \frac{1}{2^n}\text{length}(I_0)$ a subsequence x_{k_n} of x_n with $x_{k_n} \in I_n \forall n \in \mathbb{N}$

6. I_n are nested with $b_n - a_n \to 0$ therefore $\exists x \in \cap_{n=1}^{\infty} I_n = \{x\}$.

7. FINAL CLAIM: $x_{k_n} \to x$. This will finish the proof.

Note:-

Final Proof

8. We have $x_{k_n} \in I_n \forall n$, this implies that $a_n \leqslant x_{k_n} \leqslant b_n, \forall n$ and $a_n \leqslant x \leqslant b_n, x \in I_n, \forall n$. Because of this $|x_{k_n} - x| \leqslant |b_n - a_n|$. We are done here because $|b_n - a_n| \to 0$ and $0 \leqslant |x_{k_n} - x|$

😄

Note:-

Alternatively, for the last step we could have used squeeze if we showed that $a_n \to x$, $b_n \to x$, then by the squeeze theorem x_{k_n} also converges to x. Use MCT to show that they converge, and then Algebraic Limit Theorem.

16.4 Tuesday October 22

16.4.1 Reminders

Definition 16.4.1: Bolzano-Weierstrass Theorem (B-W) Theorem

Let (a_n) be a bounded sequence. Then a_n has a convergent subsequence.

Meaning: $\exists (a_{kn})$ subsequence such that a_{kn} converges.

Definition 16.4.2: Cauchy Sequence

A sequence (a_n) is cauchy if $\forall \epsilon > 0 \exists N \in \mathbb{N}$ such that if $n, m \geqslant N$, then $|a_n - a_m| < \epsilon$.

Any two terms after some point become arbitrarily close to one another.

Theorem 16.4.1

A sequence (a_n) is convergent $\leftrightarrow$ (a_n) is Cauchy.

Proof: $(\to)$ Assume that $a_n \to a$, and we want to show that $\forall \epsilon > 0 \exists N \in \mathbb{N}$ such that if $n, m \geqslant N$ then $|a_n - a_m| < \epsilon$. By definition of Cacuchy.

Note:-

Scratch

$$\begin{aligned} |a_n - a_m| &= |a_n - a + a - a_m| \\ &\leqslant |a_n - a| + |a_m - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} \end{aligned}$$

Let $\epsilon > 0$. Set $\epsilon' = \frac{\epsilon}{2} > 0$. Since $a_n \to a$, for this $\epsilon' > 0 \exists N \in \mathbb{N}$ such that if $n \geqslant N$ then $|a_n - a| < \frac{\epsilon}{2} = \epsilon'$ If $n, m \geqslant N$ we have: $|a_n - a_m| \leqslant |a_n - a| + |a_m - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Note:-

Other direction is harder because we need to come up with a limit.

$(\leftarrow)$ Assume that (a_n) is Cauchy. We want to show $\exists a \in \mathbb{R}$ such that $a_n \to a$.

1. The only Tool we have is BW. Thus the first step is to show that (a_n) is bounded.
   We will show that (a_n) is bounded. We know that a_n is cauchy. This means that for $\epsilon = 1 \exists N \in \mathbb{N}$ such that if $n, m \geqslant N$ then $|a_n - a_m| < 1$. In particular: we get that $|a_n - a_N| < 1$. Here we applied $m = N$. This gives us $-1 < a_n - a_N < 1$, which gives us $a_N - 1 < a_n < a_N + 1, \forall n \geqslant N$.
   Set $m = \min\{a_1, a_2, \ldots, a_{N-1}\}$ and $M = \max\{a_1, \ldots, a_{N-1}, a_{N+1}\}$, this implies that $m \leqslant a_n \leqslant M, \forall n \geqslant 1$. Thus a_n is bounded.

2. Step 2 will be to show that a_n converges. Since a_n is bounded, by BW theorem, a_n has a subsequence a_{kn} that converges to some limit a. $a_{kn} \longrightarrow a$. We will show that $a_n \longrightarrow a$. The trick is exactly similar to the forward direction. We know first that $\forall \epsilon > 0 \exists N \in \mathbb{N}$ such that if $n \geqslant N_1$, then $|a_{kn} - a| < \epsilon$. We know this because $a_{kn} \longrightarrow a$. We also know that a_n is cauchy: $\forall \epsilon > 0 \exists N_2 \in \mathbb{N}$ such that if $m, n \geqslant N_2$ then $|a_n - a_m| < \epsilon$. We want to show that $\forall \epsilon > 0 \exists N \in \mathbb{N}$ such that if $n \geqslant N$ then $|a_n - a| < \epsilon$.

3. Formal Proof $(\longleftarrow)$. Let $\epsilon > 0$. Set $\epsilon' = \frac{\epsilon}{2} > 0$. By BW applied for $\epsilon' > 0$, $\exists N \in \mathbb{N}$ such that if $n \geqslant N$ then $|a_{kn} - a| < \frac{\epsilon}{2}$. By the a_n being cauchy applied for $\epsilon' = \frac{\epsilon}{2} > 0$, $\exists N_2 \in \mathbb{N}$ such that if $n, m \geqslant N_2$ then $|a_n - a_m| < \frac{\epsilon}{2}$. Set $N = \max\{N_1, N_2\}$. Then for $n \geqslant N$ we have $|a_n - a| \leqslant |a_n - a_{k_n}| + |a_{k_n} - a|$ by the triangle inequality we get that this is less than $\frac{\epsilon}{2} + \frac{\epsilon}{2}$.

   Note:-

   To say that $|a_n - a_{kn}| < \frac{\epsilon}{2}$ we used the fact that $K_n \geqslant N$ (something proved from last week.)

   Claim is that if $n \geqslant N$ then $|a_n - a_{k_n}| < \frac{\epsilon}{2}$. Apply Cauchy for n and $m = k_n$. We are allowed to do this because $k_n \geqslant N$ which is something we proved last week.

   This implies that $a_n \longrightarrow a$ and we are done.

😄

Chapter 17

Chapter 3

17.1 Open and Closed Sets + Compact

17.1.1 Reminder

Definition 17.1.1: Epsilon Neighborhood

Given $a \in \mathbb{R}$, and $\epsilon > 0$: We define the ϵ neighborhood of a is $V_\epsilon(a) = (a - \epsilon, a + \epsilon)$.

Definition 17.1.2: Open Sets

A set $O \subseteq \mathbb{R}$ is called open if for every $a \in O$ there exists $\epsilon > 0$ such that $V_\epsilon(a) \subseteq O$.

17.1.2 Determine Which of the following sets are open

Note:-
Open sets are more general than open intervals.

Example 17.1.1 (Example 1)
(a, b), the open interval. We can probably guess that this is open.

Solution: Yes. Let $x \in (a, b)$. Pick any $\epsilon < \min\{x - a, b - x\}$.

Example 17.1.2 (Example 2)
$[a, b]$

Solution: Not open because $x = a$, $\forall \epsilon > 0, x - a \notin O$. No matter what ϵ I take, it will intersect the complement of this set.

Example 17.1.3 (Example 3)
$A = (0, 1) \cup \{2\}$

Solution: Not Open.Similar to 2. The bad point now is 2. The argument is exactly the same as in 2. The epsilon neighborhood of 2 $\forall \epsilon > 0$ intersects the complement of A

Example 17.1.4 (Example 4)
$B = (0, 1) \cup (2, 3)$

Solution: Yes. Open for the same reason as 1. If $x \in (0,1) \cup (2,3)$ and for each one of these we can argue in the same way as in 1.

Example 17.1.5 (Example 5)

$A = \mathbb{Q}$

Solution: Not open. Let $q \in \mathbb{Q}$, and $\epsilon > 0$. This implies that $V_\epsilon = (q-\epsilon, q+\epsilon)$. Note that $q-\epsilon < q+\epsilon$. By the density of $\mathbb{I} = \mathbb{R} - \mathbb{Q}$ we know that $\exists r \in \mathbb{I}$ such that $q-\epsilon < r < q+\epsilon \to r \in V_\epsilon(q), r \notin \mathbb{Q} \to$ any ϵ neighborhood intersects the complement of $\mathbb{Q}$.

Example 17.1.6 (Example 6)

$\mathbb{R}$

Solution: Yes. Open by definition nothing really to show here.

Theorem 17.1.1 Properties of Open Sets

1. The union of an arbitrary collection of open sets is open.
2. The intersection of finitely many open sets is open.

Note:-

Pay special attention to the wording of arbitrary collection of open sets and finitely many open sets

Proof:

1. Let $\{U_i\}_{i\in I}$ be a collection of sets.

   Note:-

   I = index set, I can be finite countable or uncountable.

   We want to show $U = \cup_{i\in I} U_i$ is open. Let $x \in U$. Because U is the union of the U_i we know that $x \in U_i$ for some $i \in I$. Since U_i is open, $\exists \epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subseteq U_i \subseteq U$. This implies that $V_\epsilon(x) \subseteq U$. Done.

2. Let $U_1, U_2, \ldots, U_n$ be a finite collection of open sets. We want to show that if we now set $U = \cap_{i=1}^n U_i$ is open. Let $x \in U \to x \in U_i$ for all $i = 1, \ldots, n$. Each U_i is open which means that for each $i \in \{1, \ldots, n\}$ we can find some $\epsilon_i > 0$ such that $(x-\epsilon_i, x+\epsilon_i) \subseteq U_i$, by assumption. Now we want to show that $\exists \epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subseteq U$. Pick $\epsilon = \min\{e_1, e_2, \ldots, e_n\}$ which must be a positive number. $\epsilon \leqslant \epsilon_i$ for $i = 1, \ldots, n$; We have $(x-\epsilon, x+\epsilon) \subseteq (x-\epsilon_i, x+\epsilon_i) \subseteq U_i$. which means that $(x-\epsilon, x+\epsilon) \subseteq U_i$ for all i this implies that $(x-\epsilon, x+\epsilon) \subseteq \cap_{i=1}^n u_i = u$.

☺

Question 27

if we had infinitely many $\{u_i\}_i \in I$ infinitely many open sets and we had $x \in \cap_{i\in I} U_i$. We can take $\epsilon inf\{\epsilon_i\}$. Problem is that ϵ can be 0. The counter example is $U_n = (-\frac{1}{n}, \frac{1}{n}), n \in \mathbb{N}$. This implies that $\cap_{n=1}^\infty U_n = \{0\}$ which is not open.

Note:-

Closed is not equivalent to not open.

Definition 17.1.3: Closed Set

A set $F \subseteq \mathbb{R}$ is called closed if it's complement is open.

Note:-

The $\emptyset, \mathbb{R}$ are both open and closed.

Note:-

$\mathbb{Q}$ is neither open nore closed, Similarly $\mathbb{I}$ is neither open nor closed.

Question 28: Why do we define closed this way?

17.2 Thursday October 24

17.2.1 Reminders

1. A set $U \subseteq \mathbb{R}$ is called open if $\forall x \in U \exists \epsilon -$neighborhood $V_\epsilon(x) = (x-\epsilon, x+\epsilon)$ such that $V_\epsilon(x) \subseteq U$.
2. Examples of this include $(a,b), \cup(c,d), \mathbb{R}, \emptyset$
3. Non-Examples: $\mathbb{Q}, [a,b], (0,1) \cup \{2\}$
4. A set $F \subseteq \mathbb{R}$ is closed if F^c is open
5. Note that open and closed may be mutually exclusive in english but not necessarily in this class. Consider that $\mathbb{R}, \emptyset$ are both open and closed. While $\mathbb{Q}$ is neither.

Example 17.2.1 (Warm-Up)
Determine which of the following sets are closed:

1. $[a,b]$ Closed. Closed interval, It's complement is the union of $(-\infty, a) \cup (b, \infty)$
2. $[a,b)$ Not Closed. The complement contains b, so the complement intersects the set we are trying to look at.
3. $[a,\infty)$ Yes.
4. $[0,1] \cup \{2\}$ Yes.
5. $\mathbb{Q}$ No. Same reason why it is not open. $\mathbb{Q}^c = \mathbb{I}$ if $r \in \mathbb{I}$, then $\forall \epsilon > 0 (r-\epsilon, r+\epsilon)$ contains rational numbers by density of $\mathbb{Q}$.
6. $\mathbb{Q}$ and $[a,b)$ are examples of neither open nor closed intervals.

17.2.2 Properties of open sets

1. Arbitrary unions of open sets are open.
2. Finite intersections of open sets are open.

17.2.3 Properties of closed sets

1. Arbitrary intersections of closed sets are closed i.e. if $\{F_i\}_{i\in I}$ is a family of closed sets, then $\cap_{i\in I} F_i$ is closed.
2. Finite Unions of closed sets are closed. Meaning if $F_1, \ldots, F_n$ are closed sets, then $\cup_{i=1}^n F_i$ is closed.

Note:-

Proof of these are left as an exercise. The main point is that we can apply what we know about open sets using deMorgan's Laws.

Example 17.2.2

$(\cap_{i \in I} F_i)^c = \cup_{i \in I} F_i^c \longrightarrow \cap_{i \in I} F_i$ closed.

Question 29: Why do we give this definition of a closed set?

Solution: Hope to justify by the end of the day.

Definition 17.2.1

Let $A \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is called a limit point of A, accumulation point or cluster point, if:

$\forall \epsilon > 0$ the $\epsilon-$neighborhood of x intersects A in some point other than x.

Equivalently: x is a limit point of A if $\forall \epsilon > 0, [(V_\epsilon(x)) \cap (A - \{x\})] \neq \emptyset$.

Example 17.2.3

Find all the limit points of $(0, 1] \cup \{2\}$.

Note:-

Notation. If $A \subseteq \mathbb{R}$, the set of limit points of A is denoted by A'.

Solution: $A' = \{[0, 1]\}$. Make some observations.

1. A limit point of A is not necessarily an element of A
2. Not every element of A is necessarily a limit point
3. We threw 2 out, and we have to include 0.

Theorem 17.2.1

A set $F \subseteq \mathbb{R}$ is closed if and only if F contains all of its limit points.

Note:-

This is why we have our definition of closed. A closed set contains all of its limit points.

Theorem 17.2.2

Let $A \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is a limit point of A if and only if $\exists$ a sequence $(x_n) \subseteq \mathbb{R}$ such that:

1. $x_n \in A, \forall n \in \mathbb{N}$
2. $x_n \longrightarrow x$
3. $x_n \neq x, \forall n$

Note:-

If x is a limit point, we should be able to find a sequence that approaches x that is different from x. This means we can get very close to x by elements in our set.

Note:-

Note that 0 is a limit point of A from above because $x_n = \frac{1}{n} \longrightarrow 0$, $x_n \neq 0$ and $x_n \in A \forall n$.

Proof: ($\longrightarrow$) Assume x is a limit point of A. This means $\forall \epsilon > 0$, $V_\epsilon(x) \cap A - \{x\} \neq \emptyset$. Now we want to construct a sequence.

1. For $\epsilon = 1$, $V_1(x) \cap A - \{x\} \neq \emptyset \to \exists x_1 \in V_1(x) \cap A_{\{}x\}$. This means that $x_1 \in V_1(x) \to |x_1 - x| < 1$. $x_1 \in A \setminus \{x\} \to x_1 \neq x$.

2. For $\epsilon = \frac{1}{2}, V_{\frac{1}{2}}(x) \cap A \setminus \{x\} \neq \emptyset \to \exists x_2 \in V_{\frac{1}{2}}(x) \cap A \setminus \{x\}$, then $x_2 \neq x$ and $x_2 \in A \setminus \{x\}$.

3. Every time we get three pieces of information something of the form $|x_2 - x| < \frac{1}{2}$, $x_2 \neq x$ and $x_2 \in A$.

Note:-

General Step:

4. For $\epsilon = \frac{1}{n}, V_{\frac{1}{n}}(x) \cap A \setminus \{x\} \neq 0 \to \exists x_n$such that $|x_n - x| < \frac{1}{n}$ and $x_n \in A$ and $x_n \neq x$

We have constructed a sequence x_n with $x_n \in A, \forall n \in \mathbb{N}$ $x_n \neq x, \forall n \in \mathbb{N}$ and $|x_n - x| < \frac{1}{n} \to x - \frac{1}{n} < x_n < x_{\frac{1}{n}}$. By the Squeeze theorem we get that $x_n \to x$. The three properites needed were: $x_n \in A$, $x_n \neq x$ and $x_n \to x$. $(\leftarrow)$ Suppose $\exists (x_n)$ sequence satisfying:

1. $x_n \in A$ for all $n \in \mathbb{N}$

2. $x_n \neq x, \forall n \in \mathbb{N}$

3. $x_n \to x$

We want to show that x is a limit point of A, meaning, we want to show $\forall \epsilon > 0$, $V_\epsilon(x) \cap A \setminus \{x\} \neq \emptyset$ Let $\epsilon > 0$, Since $x_n \to x$, for this $\epsilon > 0$ $\exists N \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ for all $n \geqslant N$. In particular x_N satisfies $x_N \in A$ by assumption. $x_N \neq x$ again by assumption. $|x_N - x| < \epsilon \to x_N \in V_\epsilon(x)$. The first two above imply that $x_N \in A \setminus \{x\}$. So we are done, we get that $x_N \in V_\epsilon(x) \cap A \setminus \{x\}$.

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Example 17.2.4 (Previous Example)

$A = (0, 1] \cup \{2\}$:

We see again why 2 is not a limit point. If there is a sequence within our set that converges to 2, must be the constant sequence. A point like this will be called isolated points.

Definition 17.2.2: Isolated Points

Let $A \subseteq \mathbb{R}$, A point $a \in A$ is called isolated if a is not a limit point.

Question 30: Find an equivalent condition to this

Solution: a not a limit point of A means $\exists \epsilon > 0$ such that $V_\epsilon(a) \cap A \setminus \{a\} = \emptyset$.

In fact a point $a \in A$ is isolated $\leftrightarrow \exists \epsilon > 0$ such that $V_\epsilon(a) \cap A = \{a\}$.

$a \in \mathbb{R}$ not a limit point of A means $\exists \epsilon > 0$ such that $V_\epsilon(a) \cap A - \{a\} = \emptyset \to V_\epsilon(a) \cap A = \emptyset$ OR $\{a\}$.

But isolated points are always elements in the set which means we only have $\{a\}$.

Question 31: Closed Set containing only isolated points?

Solution: The natural numbers: $\mathbb{N}$. There exist infinite sets containing only isolated points.

17.2.4 Proof of Theorem 2

Theorem 17.2.3

A set $F \subseteq \mathbb{R}$ is closed $\leftrightarrow$ F containts all its limit points.

Proof: ($\longrightarrow$) Suppose $F \subseteq \mathbb{R}$ is closed. This means that F^c is open. Let $x \in \mathbb{R}$ be a limit point of F. We want to show that $x \in F$. Suppose for contradiction that $x \notin F$. This means that $x \in F^c$. Which is open, meaning $\exists \epsilon > 0$ such that $V_\epsilon(x) \subseteq F^c$. Which implies that $V_\epsilon(x) \cap F = \emptyset \longrightarrow x$ is not a limit point, which is our contradiction.
($\longleftarrow$) Suppose F contains all its limit points. We want to show that F^c is open. To show something is open, follow the definition. Let $x \in U = F^c$. $x \in F^c \longrightarrow x \notin F$. Since F contains all its limit points, we get that x is not a limit point of F. This means by definition that $\exists \epsilon > 0$ such that $V_\epsilon(x) \cap F \setminus \{x\} = \emptyset$ and $x \notin F$. These two facts together imply that $V_\epsilon(x) \cap F = \emptyset \longrightarrow V_\epsilon(x) \subseteq F^c \longrightarrow F^c$ is open, and therefore F is closed.

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Definition 17.2.3: Compact Set

A set $K \subseteq \mathbb{R}$ is called compact if every sequence $(x_n) \subseteq K$ has a subsequence x_{k_n} has a subsequence (x_{k_n}) that converges to some element in k.

K compact $\leftrightarrow$ for every $(x_n) \subseteq K \exists$ subsequence $x_{k_n} \longrightarrow x \in K$

Example 17.2.5 (Which of these sets are Compact)

1. $(0, 1]$ No. $\frac{1}{n}$, converges to something but the something is not in the set.
2. $[1, \infty)$ Not be compact. $x_n = n$ is contained in the set, but has no convergence subsequence.
3. $[a, b]$ Yes. Bounded, every sequence bounded, every subsequence converges. Used BW, and closed. Also sequence is closed, so if contains its limit points.
4. $\mathbb{Q}$ No.

17.3 Tuesday October 29th

Note:-

No classes next Tuesday. Quiz tomorrow. Midterm 2 on Tuesday November 19th.

17.3.1 Reminders

1. A set $U \subseteq \mathbb{R}$ is open if $\forall x \in U \exists \epsilon > 0$ such that $V_\epsilon(x) \subseteq U$
2. A set $F \subseteq \mathbb{R}$ is closed if and only if F^c is open if and only if F contains all of its limit points
3. Let $A \subseteq \mathbb{R}$ A point $x \in \mathbb{R}$ is called a limit point of A if $\forall \epsilon > 0 V_\epsilon(x) \cap A \setminus \{x\} \neq \emptyset$

Note:-

Quiz tomorrow, two sets need to determine if they are open or closed or neither.

Definition 17.3.1: Compact

A set $K \subseteq \mathbb{R}$ is called compact if every sequence $(x_n) \subseteq K$ has a convergent subsequence$x_{k_n} \longrightarrow x \in K$.

Theorem 17.3.1

$K \subseteq \mathbb{R}$ is compact $\leftrightarrow$ K is closed and bounded

Proof: $(\leftarrow)$: Let $(x_n) \subseteq K$. Since K is bounded, $\rightarrow (x_n)$ is bounded. By BW, $\exists$ a subsequence $(x_{k_n}) \rightarrow x \in \mathbb{R}$. Since K is closed, $x \in K$.

$(\rightarrow)$: Suppose K is compact. We want to show it is closed and bounded. First we show that K is closed. Enough to show that K contains all of its limit points. Let $x \in \mathbb{R}$ be a limit point of K. From last time, we proved $\exists (x_n)$ sequence such that:

1. $x_n \in K, \forall n$
2. $x_n \rightarrow x$
3. $x_n \neq x, \forall n$

Since (x_n) is a sequence in K and K is compact, x_n has a convergent subsequence $x_{k_n} \rightarrow y \in K$. Since the original sequence, $x_n \rightarrow x$, it must be the case that $x_{k_n} \rightarrow y = x \rightarrow x \in K$. Which shows that K is closed. Show K is bounded. Suppose for contradiction that K is not bounded. Without loss of generality, let us assume that K is not bounded above. This implies that 1 is not an upper bound of $K \rightarrow \exists x_1 \in K$ with $x_1 > 1$. Similarly, 2 is not an upper bound of $K \rightarrow \exists x_2 \in K$ such that $x_2 > 2$. Proceed inductively, for every $n \in \mathbb{N}$, because n is not an upper bound of K, $\exists x_n \in K$ such that $x_n > n$, this implies the sequence $(x_n) \subseteq K$ and $x_n \rightarrow \infty \rightarrow x_n$ has no convergent subsequences. This is a contradiction to the compactness of K.

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Chapter 18

Chapter 4: Limits of functions + continuity

Suppose $f : A \to \mathbb{R}$

Question 32: What does it mean $\lim_{x\to c} f(x) = L$?

Solution: $\lim_{x\to c}$, this process only makes sense if c is a limit point of our set. We need to be able to approach c from very close points in the domain. We will only consider limits $\lim_{x\to c} f(x)$ for c a limit point of A.

18.0.1 Reminders

Definition 18.0.1: Limit Point

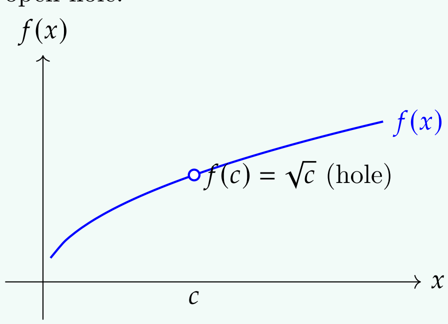
Let $A \subseteq \mathbb{R}$ a point $c \in A$ is a limit point of A if $\forall \epsilon > 0$, $V_\epsilon(c) \cap A \setminus \{c\} \neq \emptyset \leftrightarrow$
$\leftrightarrow \exists (x_n) \subseteq A$ with $x_n \to c, x_n \neq c$.

Definition 18.0.2: $\epsilon\delta$ Definition of a limit of a function

Let $f : A \to \mathbb{R}$ a function and $c \in \mathbb{R}$ a limit point of A. We say that $\lim_{x\to c} f(x) = L$ if: $\forall \epsilon > 0 \exists \delta > 0$ such that if $x \in A$ with $0 < |x - c| < \delta$ then $|f(x) - L| < \epsilon$.

Example 18.0.1 (Example 1)

What do we expect the limit of $f(x)$ to be as we approach c? We expect the limit to be $f(c)$ despite the open hole.



Example 18.0.2 (Example 3 from class)

Can't find $\delta > 0$ that does the job this is becuase no matter which $\delta > 0$ we take, if $x \in (c, c+\delta), f(x) \notin V_\epsilon(L)$.

Note:-

Take away: $\epsilon > 0$ represents the distance from $L = \lim_{x\to c} f(x)$, this is a distance in the y-axis. Given $\epsilon > 0$ we look for $\delta > 0$ which is the distance from c on the x-axis.

Definition 18.0.3: $\epsilon\delta$ definition using epsilon neighborhoods

We say that $\lim_{x\to c} f(x) = L$ if: $\forall \epsilon > 0 \exists \delta > 0$ such that for $x \in A$ and $x \neq c$, $x \in V_\delta(c)$ then $f(x) \in V_\epsilon(L)$.

18.0.2 Examples

Example 18.0.3 (Proof with $\epsilon\delta$)

1. $\lim_{x\to 2}(-5x+1) = 9$: Linear $\epsilon\delta$
2. $\lim_{x\to -3} 5x^2 = 45$: Quadratic $\epsilon\delta$

Note:-

Linear ones are super easy. If we do 2 of them we can do all of them. Definitely going to be on the midterm.

Proof: Linear $\epsilon\delta$ above. $\lim_{x\to 2}(-5x+1) = -9$
$f(x) = -5x+1$, $c = 2, L = -9$. We want to show that $\forall \epsilon > 0 \exists \delta > 0$ such that if $0 < |x-2| < \delta$ then $|f(x)+9| < \epsilon$.

Note:-

Scratch Below: We want to make $|f(x)+9| < \epsilon$. Simplify this as much as we can possibly do. Hopefully this will lead us to which delta we have to take.

$|f(x)+9| = |-5x+1+9| = |-5x+10| = |(-5)(x-2)|$
The $x-2$ is supposed to appear.
$= 5 \cdot |x-2|$
We want to make $|-5x+10| < \epsilon$
$= 5|x-2| < \epsilon$, if we make $|x-2| < \frac{\epsilon}{5}$ we are done. This means that $\delta = \frac{\epsilon}{5}$.

Note:-

Formal Below

Let $\epsilon > 0$. Set $\delta = \frac{\epsilon}{5} > 0$. If $0 < |x-2| < \delta$ we have: $|f(x)+9| = |-5x+10|$
$= 5|x-2| < 5\delta = 5\frac{\epsilon}{5} = \epsilon$.

Note:-

$f(x) = ax+b$ let $\epsilon > 0$ what δ should we take? Take $\delta = \frac{\epsilon}{|a|}$

😄

Proof: $\lim_{x\to -3} 5x^2 = 45$
$f(x) = 5x^2, c = -3, L = 45$
We want to show $\forall \epsilon > 0 \exists \delta > 0$ such that if $0 < |x+3| < \delta$ then $|f(x)-45| < \epsilon$

Note:-

Scratch Below

$|f(x)-45| =$
$|5x^2-45| = 5|x^2-9|$

$$= 5|x-3||x+3|$$

Note:-

$x-3$ will appear

We want to make $5|x-3||x+3| < \epsilon$ Note that δ can never depend on x. This means we can not proceed in the same manner as last time.

Note:-

Strategy: $5|x-3||x+3|$ the limit we take is at -3, so the relevant piece is $|x+3|$, while the irrelevant piece is $|x-3|$. The strategy is to bound the irrelevant piece. Suppose that we can make $|x-3| \leqslant M$. Then $5|x-3||x+3| \leqslant 5M|x+3| < \epsilon$
$\delta = \frac{\epsilon}{5M}$

Note:-

If we find $\delta > 0$ that works, any smaller δ' will also work. Because of this we may assume that $\delta \leqslant 1$. If $|x+3| < \delta$ then $|x+3| < 1$.

$|x-3| = |x+3-6| \leqslant |x+3| + |-6|$
$|x-3| < 7$

Then $5|x-3||x+3| \leqslant 5 \cdot 7|x+3|$ we want to make $35|x+3| < \epsilon$. We can take $\delta = \min\{1, \frac{\epsilon}{35}\}$

Note:-

Formal Proof

Let $\epsilon > 0$. Choose $\delta = \min\{1, \frac{\epsilon}{35}\}$ If $0 < |x+3| < \delta$ then:

1. $\delta \leqslant 1 \rightarrow$
   $|x-3| \leqslant |x+3| + 6 < 1 + 6 = 7$
2. $|f(x) - 45| = 5|x-3||x+3|$
   $< 35|x+3|$
   $< 35\delta \leqslant 35\frac{\epsilon}{35} = \epsilon.$

😊

Note:-

This proof allows us to work with sequences instead of $\epsilon\delta$.

18.1 Thursday October 31st

18.1.1 Reminders

Note:-

HW 9 is due Monday

Definition 18.1.1

$f : A \rightarrow \mathbb{R}$ $c \in \mathbb{R}$ a limit point of A. Then $\lim_{x\to c} f(x) = L$ if $\forall\epsilon > 0 \exists\delta > 0$ such that if $x \in A$ with $0 < |x-c| < \delta$ then $|f(x) - L| < \epsilon$.

Example 18.1.1 (Quadratic Example)

$$\lim_{x\to -1}(x^2+2) = 3$$
$$|x^2-1|$$
$$|x-1||x+1|$$

The $|x-1|$ part is irrelevant, while the $|x+1|$ part is relevant. The part that is relevant is the $|x-c|$ part.

Note:-

The strategy is to bound the irrelvant part. The trick with δ is to assume that it is less than or equal to one.

Assume $\delta \leqslant 1$. We will rerember this until the very end. If $|x+1| < \delta \leqslant 1 \to |x-1 = |x+1-2||$

$$\leqslant |x+1| + |-2| < \delta + 2$$
$$\leqslant 1 + \underbrace{2}_{M>0} = 3$$
$$= |x-1| < 3$$
$$|f(x) - L| = |x-1||x+1| <$$
$$< 3|x+1| < \epsilon$$
$$\text{Take } \delta \leqslant \frac{\epsilon}{3}$$

Note:-

Formal Below:

Proof: Want to show $\forall \epsilon 0 \exists \delta > 0$ such that if $0 < |x+1| < \delta|$ then $|f(x) - 3| < \epsilon$.
Let $\epsilon > 0$ (fixed). Choose $\delta = \min\{1, \frac{\epsilon}{3}\} > 0$. If $0 < |x+1| < \delta$ we have:
$|x-1| \leqslant |x+1| + 2 < \delta + 2 \leqslant 3$ because $\delta \leqslant 1$.
$|f(x) - 3| = |x-1||x+1| < 3|x+1| <$
$3\delta \leqslant 3\frac{\epsilon}{3} = \delta$.

😊

Question 33: Why do we need $0 < |x-c| < \delta$ in the definition

Solution: Consider the graph from yesterday with a hole. If we define the function to be $f(x) \neq L$ then the definition without < 0, still works.

Theorem 18.1.1 Sequential Criterion For Functional Limits

Given a funciton $f : A \to \mathbb{R}$ with $c \in \mathbb{R}$ a limit point of A, $\lim_{x \to c}(f(x)) = L \leftrightarrow \forall$ sequence $(x_n) \subseteq A$ with $x_n \to c, x_n \neq c \forall n$ it follows $f(x_n) \to L$.

1. $\lim_{x \to c} f(x) = L, \forall \epsilon > 0 \exists \delta > 0$ such that if $x \in A$ with $0 < |x-c| < \delta, |f(x) - L| < \epsilon$
2. $\lim_{x \to c} f(x) = L \leftrightarrow \forall$ sequences $(x_n) \subseteq A$ with $x_n \to c, x_n \neq c, \forall n$ it follows that $f(x_n) \to L$

Proof: Forward direction is a bit easier becuase we can assume the L. We start with the forward direction.
($\to$) We know that 1 above is true. And we want to show $\forall (x_n) \subseteq A$ with $x_n \to c, x_n \neq c, f(x_n) \to L$. Let $(x_n) \subseteq A$ be a sequence such that $x_n \to c, x_n \neq c, \forall n$. We want to show that $f(x_n) \to L$.
Here we go with the definition of convergence. We want to show that $\forall \epsilon > 0 \exists N \in \mathbb{N}$ such that if $n \geqslant N$ then $|f(x_n) - L| < \epsilon$. Let this be our goal.
Fix $\epsilon > 0$ arbitrary but fixed. Since $\lim_{x \to c} f(x) = L$ for this $\epsilon > 0$ we can find $\delta > 0$ such that if $x \in A$ with $0 < |x-c| < \delta$ it follows that $|f(x) - L| < \epsilon$. We also know that (x_n) is contained in A, it approaces c and it is not equal to c.
Since $(x_n) \to c$, for the $\delta > 0$ we can find $N \in \mathbb{N}$ such that if $n \geqslant N$, $|x_n - c| < \delta$ Here we applied the epsilon definition for convergence of a sequence for $\epsilon' = \delta$.
For the sequence (x_n) we know:

1. $x_n \in A, \forall n$
2. $x_n \neq c, \forall n \to 0 < |x_n - c|, \forall n \in \mathbb{N}$
3. If $n \geqslant N, |x_n - c| < \delta$

This implies that $x_n \in A$ and for $n \geqslant N$, $0 < |x_n - c| < \delta$. Now we can immedaitely see that $|f(x_n) - L| < \epsilon$. Thus $f(x_n) \to L$.

Note:-

Have to do something similar for a problem on the homework.

($\longleftarrow$) For the other direction Assume $\forall(x_n) \subseteq A$ with $x_n \longrightarrow c, x_n \neq c \forall n, f(x_n) \longrightarrow L$. And we want to show: $\lim_{x\to c} f(x) = L$. We proceed by contradiction. Assume $\lim_{x\to c} f(x) \neq L$.

See below for the negation of the $\epsilon\delta$ definition of a limit. This means $\exists\epsilon > 0$ such that $\forall\delta > 0 \exists x_\delta \in A$ with $0 < |x_\delta - c| < \delta$ and $|f(x_\delta) - L| \geqslant \epsilon$. Apply this for $\delta = \frac{1}{n}$. For each $n \in \mathbb{N}$, take $\delta = \frac{1}{n} > 0$. Then $\exists x_n \in A$ with $0 < |x_n - c| < \frac{1}{n}$ and $|f(x_n) - L| \geqslant \epsilon$.

For the sequence (x_n) we got:

1. $x_n \in A, \forall n$
2. $x_n \neq c, \forall n$ since $0 < |x_n - c|$
3. $x_n \longrightarrow c$ (For example by squeeze).
4. $f(x_n) \not\rightarrow L$ does not approach L.

Note:-

Construction of (x_n)

1. Apply $\exists\epsilon > 0$ such that $\forall\delta > 0 \exists x_\delta \in A$ with $0 < |x_\delta - c| < \delta$ and $|f(x_\delta) - L| \geqslant \epsilon$ for $\delta = 1 \longrightarrow \exists x_1 \in A$ with $0 < |x_1 - c| < 1$ and $|f(x_1) - L| \geqslant \epsilon$.
2. Apply the definition above again for $\delta = \frac{1}{2} > 0$ this implies that $\exists x_2 \in A$ with $0 < |x_2 - c| < \frac{1}{2}$ and $|f(x_2) - L| \geqslant \epsilon$. Continue in this manner inductively for each $n \in \mathbb{N}$. $\exists x_n \in A$ such that $0 < |x_n - c| < \frac{1}{n}$ and $|f(x_\delta) - L| \geqslant \epsilon$.

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Question 34: What is the negation of the $\epsilon\delta$ Negation?

Solution: $\lim_{x\to c} f(x) \neq L$ if: $\exists\epsilon > 0 \forall\delta > 0$ such that if $x \in A$ with $0 < |x - c| < \delta$, then $|f(x) - L| \geqslant \epsilon$.
Class Answer: ***Solution:*** $\exists\epsilon > 0$ such that $\forall\delta > 0 \exists x_\delta \in A$ with $0 < |x_\delta - c|\delta$ and $|f(x_\delta) - L| \geqslant \epsilon$

Note:-

This proof allows us to work with sequences instead of $\epsilon\delta$.

18.1.2 Applications

Example 18.1.2 (Algebra of limits of functions)

Let $f, g : A \longrightarrow \mathbb{R}$, $c \in \mathbb{R}$ a limit point of A. Suppose $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = L_2$ then:

1. $\lim_{x\to c}(f(x) + g(x)) = L_1 + L_2$
2. $\lim_{x\to c} f(x)g(x) = L_1 L_2$
3. if $L_2 \neq 0$ then $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}$.

Proof: Proof of 1 above. By the sequential Criterion it is enough to show $\forall(x_n) \subseteq A$ with $x_n \longrightarrow c$ and $x_n \neq c$, $f(x_n) + g(x_n) \longrightarrow L_1 + L_2$.

Since $\lim_{x\to c} f(x) = L_1$ it follows that $f(x_n) \longrightarrow L_1$ and similarly $g(x_n) \longrightarrow L_2$. Together this implies that $f(x_n) + g(x_n) \longrightarrow L_1 + L_2$.

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Example 18.1.3 (Application 2)
Use sequential Criterion to show that certain limits don't exist.

Say we have $f : A \to \mathbb{R}$, $c \in \mathbb{R}$ a limit point of A. We want to show that $\lim_{x\to c} f(x)$ DNE.

One possible method is to find a sequence $(x_n) \subseteq A$ with $x_n \to c, x_n \neq c$ and $f(x_n)$ diverges.

Another possibility is to find two sequences $(x_n), (y_n)$ both approaching c but $f(x_n)_{n\to c} \neq f(y_n)_{n\to c}$ Note that x_n as $n \to c = y_n$.

Example 18.1.4
$f : (0, \infty) \to \mathbb{R}$. $c = 0, f(x) = \sin(\frac{1}{x})$. Convince me that $\lim_{x\to 0} f(x)$ DNE.

Solution: $\frac{1}{n} \to 0$ and $f(\frac{1}{n}) = \sin(n)$. Hard to convince/motivate.

$x_n = \frac{1}{2\pi n} \to 0$ and $\sin(\frac{1}{x_n}) = \sin(2\pi n) = 0$. This means we found a sequence $(x_n) \subseteq (0, \infty)$ with $x_n \to 0$ and $f(x_n) \to 0$. Take $y_n = \frac{1}{2\pi n} + \frac{\pi}{2} \to 0$ and $f(y_n) = 1 \to 1$.

$\lim_{x\to 0} \sin(\frac{1}{x})$ DNE.

Instead we could use one sequence $a_n = \frac{1}{\pi n + \frac{\pi}{2}} = \frac{2}{\pi n}$.

Note:-
We can conclude out of this example that this limit is so bad that we can find sequences converging to zero, where the f's can go to any possible number between $-1, 1$.

18.2 Thursday November 7th

18.2.1 Reminders

1. Midterm on Tuesday the 19th
2. Office Hours next week regularly and Monday before the test
3. Practice Problems Released During the Weekend
4. Material:
   (a) Up to Intermediate Value Theorem (Proof not included)
   (b) This is Thursday November 7th Lecture
5. Office Hours next week
6. Extra office hours on Monday 18th November
7. No Homework Next Week

18.2.2 Reminders

1. $f : A \to \mathbb{R}$, a function $c \in \mathbb{R}$ a limit point of A. $\lim_{x\to c} f(x) = L$ means $\forall \epsilon > 0 \exists \delta > 0$ such that if $x \in A$ with $0 < |x - c| < \delta$ then $|f(x) - L| < \epsilon$.

18.3 Continuity

Definition 18.3.1

Let $f : A \to \mathbb{R}$, $c \in A$. We say that f is continuous at c if: $\forall \epsilon > 0 \exists \delta > 0$s such that if $x \in A$ with $|x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$

Note:-

For $x = c$, we always have $|f(x) - f(c)| < \epsilon$.

Question 35: What are the differences between continuity and limit?

Solution:

1. For continuity we include $x = c$ and we require c to be in the domain.
2. $L = f(c)$
3. c is allowed to be an isolated point in the domain.

Question 36: What are the similarities

Solution:

1. Continuity at c essentially says $\lim_{x \to c} f(x) = f(c)$
   We say essentialy because c doesn't have to be a limit point. See differences.
2. If $c \in A$ is a limit point of A then this is equivalent to continuity meaninf f is continuous at $c \leftrightarrow \lim_{x \to c} f(x) = f(c)$

Note:-

One of the problems in this week's HW says that if c is an isolated point, $\to$ f is continuous at c, and we don't need to check.

Note:-

We say taht f is continuous without specifying the point if f is continuous at every $c \in A$.

Theorem 18.3.1 Sequential Criterion

Let $f : A \to \mathbb{R}, c \in A$. Then f is continuous at $c \leftrightarrow \forall$ sequences $(x_n) \subseteq A$ with $x_n \to c$, then it follows that $f(x_n) \to f(c)$

Proof: Proof is omitted. Exactly the same as the proof for limits.

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18.3.1 Applications

1. Criterion for discontinuity, f will be discontinuos at $c \in A$ if we can find sequence $(x_n) \subseteq A$ with $x_n \to c$ and $f(x_n) \to f(c)$

Example 18.3.1

Let $f(x) = 1, x \in \mathbb{Q}$ and $0, x \notin \mathbb{Q}$ This is a divided function we have mentioned before, thinkin (Dirichlect function ?)

Claim f is discontinuous at every $c \in \mathbb{R}$.

Proof: Let $c \in \mathbb{R}$ we want to show f is not continuous at c.

Case 1: Suppose $c \in \mathbb{Q}$, then $f(c) = 1$. By density of $\mathbb{I} \in \mathbb{R}$ we can find a sequence $(x_n) \subseteq \mathbb{I}$ with $x_n \to c$.
$x_n \notin \mathbb{Q} \to f(x_n) = 0, \forall n$
$\to f(x_n) \to 0$ But $f(c) = 1$ so $x_n \to c$ but $f(x_n) \nrightarrow f(c)$.
Case 2: $c \notin \mathbb{Q}$ ... Left as practice. Argument is symmetric.

Note:-
The HW has a similar problem but the cases are not symmetric.

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Note:-
Application Number 2

Example 18.3.2 (Algebra of Continuous Functions)
Let $f, g : A \to \mathbb{R}$ continuous at $c \in A$. Then:

1. $f + g$ is continuous at c
2. $f \cdot g$ is continuous at c
3. $\frac{f}{g}$ is continuous at c provided that $g(c) \neq 0$.

Example 18.3.3
Every Polynomial function is continuous.

Proof: Let $p(x) = a_n x^n + \ldots + a_1 x + a_0, a_i \in \mathbb{R}$. It is enough to show that $a_i x^i$ is continuous, because then we can use 1 above.
We can break it up more becuase $a_i x^i = a_i(x \ldots x)$
Then it is enough to sho wthat the functions $f(x) = a =$ constant and $g(x) = x$ are continuous. $\leftarrow$ Very easy to show and left as practice

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Example 18.3.4
Let $f(x) = x\sin(\frac{1}{x}), x \neq 0$ and $f(x) = 0$ if $x = 0$.

Claim: f is continuous at $c = 0$. Proceed with an $\epsilon\delta$ proof.

Note:-
Scratch Below

Want to show: $\forall \epsilon > 0 \exists \delta > 0$ such that if $|x| < \delta$ then $|x\sin(\frac{1}{x})| < \epsilon$
Recall $\forall y \in \mathbb{R}, |\sin(y) \leqslant 1|$ Apply this for $y = \frac{1}{x}$.
$|x\sin(\frac{1}{x})| = |x||\sin(\frac{1}{x})| \leqslant |x| < \delta$ take $\delta = \epsilon$ works! $< \epsilon$

Note:-
Formal Below

Let $\epsilon > 0$. Take $\delta = \epsilon$. Then if $x \in \mathbb{R}$ and $x \neq 0$ and $|x| < \delta$, we have:
$|x\sin(\frac{1}{x})| = |x||\sin(\frac{1}{x})| \leqslant |x| < \delta = \epsilon$.
This implies that $\forall x \in \mathbb{R}$ with $x \neq 0$ and $|x| < \delta$, $|f(x) - 0| < \epsilon$. But this is also true for $x = 0$ because $f(0) = 0$. We can conclude here.

Theorem 18.3.2 Composition of Continuous Functions

Let $f : A \to \mathbb{R}$, and $g : B \to \mathbb{R}$ functions such that $f(A) \subseteq B$. This implies that $g \circ f$ is defined.

If f is continuous at c and g is continuous at $f(c)$ then $g \circ f$ is continuous at c.

Proof: Proof in this week's Homework.

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18.3.2 Questions

With the following Assumptions:

1. Let $f : A \to \mathbb{R}$ continuous
2. Let $B \subseteq A$
3. $f(B) = \{y = f(x) : x \in B\}$

1. Suppose B is open, then $f(B)$ is open
   Solution: No. Counter Example: Any constant function does the job. $f : (0,1) \to \mathbb{R}$. Take $f(x) = 2$. $f((0,1)) = \{2\}$
2. Suppse B is closed $\to f(B)$ is closed?
   Solution: No. Counter Example: $f : [1, \infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$

Note:-

If you have $f : \mathbb{R} \to \mathbb{R}$ and $B \subseteq \mathbb{R}$ what happens for $f^{-1}(B)$. The answer then will be yes to both questions above.

18.3.3 Reminders

Definition 18.3.2: Compact Set

$K \subseteq \mathbb{R}$ is compact $\leftrightarrow$ K is closed and bounded $\leftrightarrow \forall (x_n) \subseteq K$ has a subsequence $x_{k_n} \to x \in K$ (Sequential Compactness).

Theorem 18.3.3

Let $f : K \to \mathbb{R}$ with K compact. Then $f(K)$ is compact.

Proof: We want to show that $f(K)$ is compact. This means that we want to show $\forall y_n$ (using y for $f(x)$ because $f(x)$ lives on the y axis)
$\forall (y_n) \subseteq f(K)$ there is a subsequence $y_{k_n} \to y$ with $y \in f(K$.
Let $(y_n) \subseteq f(K)$ be a sequence. $f(K) = \{y = f(x) : x \in K\}$ by definition. Then each y_n can be written as $f(x_n)$ for some x_n in K.
For every $n \in \mathbb{N}$, $y_n = f(x_n)$ for some $x_n \in K$. This way we get a sequence (x_n) contained in K. Since K is compact, x_n has a subsequence $x_{k_n} \to x$ for some $x \in K$. Since $x \in K$, f is continuous at $x \to$ by the sequential criterion for continuity: $f(x_{k_n}) \to f(x)$.
Now we are done becuase $f(x_{k_n}) = y_{k_n} \to f(x) \in f(K)$ because $x \in K$. Thus $f(K)$ is compact.

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Note:-

The above theorem we will find is a key to proving larger theorems

Theorem 18.3.4 Extreme Value Theorem (EVT)

Let $f : K \to \mathbb{R}$ continuous with K compact.

Then f attains a maximum and minimum value, i.e. $\exists x_0, x_1 \in K$ such that $f(x_0) \leqslant f(x) \leqslant f(x_1), \forall x \in K$. In this case $f(x_0)$ is the minimum value and $f(x_1)$ is the maximum value.

Proof: Theorem 1 implies that $f(K)$ is compact. Compactness first implies that $f(K)$ is bounded. Bounded means that $s = \sup f(K)$ and $t = \inf f(K)$ exist.

Now we want to show that these are actually values. The supremum and infimum are values.

Note:-

Note that L is the set of all limit points for a given set.

Note that in Homework 8, we proved that if $A \subseteq \mathbb{R}$ is bounded, then $\sup A, \inf A \in A \cup L$ for $A \cup L$ the closure of A.

This implies that $s, t \in A \cup L$. Becuase $f(K)$ is closed itself, we know that $f(K) \cup L = f(K)$

This implies that $s, t \in f(K) \to \exists x_0, x_1 \in K$ such that $s = \sup f(k) = f(x_1)$ and $t = \inf f(k) = f(x_0)$. This implies:

$f(x_0) \leqslant f(x) \leqslant f(x_1) \forall x \in K$ and we can conclude here.

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Theorem 18.3.5 Intermediate Value Theorem

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. If L is any intermediate number between $f(a)$ and $f(b)$, then $\exists c \in [a, b]$ such that $f(c) = L$

Restated: If $f(a) \leqslant L \leqslant f(b)$ or $f(b) \leqslant L \leqslant f(a) \to$
$\exists c \in [a, b]$ such that $f(c) = L$

Note:-

Every Horizontal Line $y = L$ with L between $f(a)$ and $f(b)$ must intersect the graph of f at least once.

18.3.4 True False

1. Let $f : [a, b] \to \mathbb{R}$ continuous with $f(a) < 0 < f(b)$ then $\exists c \in (a, b)$ such that $f(c) = 0$.
2. Let $f : [a, b] \to \mathbb{R}$, L is a number between $f(a), f(b)$ suppose f is continuous on (a, b) then $\exists c \in [a, b]$ such that $f(c) = L$.

18.4 Tuesday November 12th

Note:-

Extra office hours next monday 1:30-3:00.

18.4.1 Reminders

1. **Theorem 18.4.1** Continuous Functions on Compact Sets

   Let $f : K \to \mathbb{R}$ continuous, with K compact. Then $f(k)$ is compact.

2.

Theorem 18.4.2 Extreme Value Theorem (EVT)

Let $f : K \to \mathbb{R}$ with K compact, then f attains a maximum and minimum value. Meaning $\exists x_1, x_2, \in K$ such that $f(x_2) \leqslant f(x) \leqslant f(x_1) \quad \forall x \in K$.

Theorem 18.4.3 Intermediate Value Theorem (IVT)

Let $f : [a,b] \to \mathbb{R}$ be continuous. Continuos on a closed interval, so two assumptions. Let L be a number between $f(a), f(b)$. Then $\exists c \in [a,b]$ such that $f(c) = L$, meaning ever intermediate value is achieved.

18.4.2 True False From Last Time

1. Let $f : [a,b] \to \mathbb{R}$ continuous with $f(a) < 0 < f(b)$ then $\exists c \in (a,b)$ such that $f(c) = 0$.
2. Let $f : [a,b] \to \mathbb{R}$, L is a number between $f(a), f(b)$ suppose f is continuous on (a,b) then $\exists c \in [a,b]$ such that $f(c) = L$.

Solution: One is True. Two is False. You can expect a similar question on the midterm. A graph is perfectly sufficient for you answer.

We can create a graph with a hole. That is discontinuous at $x = a$. With f continuous on $(a,b]$. Then we have $f(a) < 0 < f(b)$ but $\nexists c \in [a,b]$ such that $f(c) = 0$.

Note:-

Number 1 above is a special case of IVT: Let $f : [a,b] \to \mathbb{R}$ continuous suppose $f(a) < 0 < f(b)$. Then $\exists c \in (a,b)$ such that $f(c) = 0$.

We will prove this special case of IVT. If this special case is true then the IVT is true.

Theorem 18.4.4 Intermediate Value Theorem

Proof: Let L be a number between $f(a), f(b)$. If $L = f(a)$ or $L = f(b)$, then take $c = a$ or b. Done. Suppose not. Without loss of generality assume $f(a) < L < f(b)$. Consider a new function $g : [a,b] \to \mathbb{R}, g(x) = f(x) - L$. We know that g is continuous on $[a,b]$ because it is a sum of two continuous functions. Then we compute $g(a) = f(a) - L < 0$ and $g(b) = f(b) - L > 0$. So we are in the special case. By special case $\exists c \in (a,b)$ such that $g(c) = 0 \to f(c) = L$.

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Question 37: Three Major Steps

1. Special Case $f : [a,b] \to \mathbb{R}$ continuous $f(a) < 0 < f(b) \to \exists c \in (a,b)$ such that $f(c) = 0$.
2. General Case: $f : [a,b] \to \mathbb{R}$ continuous. L a number between $f(a), f(b) \to \exists c \in [a,b]$ such that $f(c) = L$.

If $L \neq f(a), f(b)$ then we can create a new function so that we can apply the special case.

Theorem 18.4.5 Proof of Special Case of IVT Using Nested Interval Property

Proof: Assumption: $f : [a,b] \to \mathbb{R}$ continuous. $f(a) < 0 < f(b)$. Set $I_0 = [a,b]$.

1. Cut I_0 in 2 halves. Set $z_0 = \frac{a+b}{2}$ = midpoint of I_0.
   (a) If $f(z_0) = 0$ take $c = z_0$ Done.
   (b) If $f(z_0) \neq 0$ and is positive, define $I_1 = [a, z_0]$
   (c) If $f(z_0) \neq 0$ and is negative define $I_1 = [z_0, b]$.

In either case we have constructed a closed interval $[a_1, b_1]$ such that $f(a_1) < 0 < f(b_1)$ and length $(I_1) = \frac{1}{2}\text{length}(I_0)$.

2. Step 2: Take $z_1 = \frac{a_1+b_1}{2}$.

   (a) If $f(z_1) = 0 \longrightarrow$ take $c = z$. Done.

   (b) If note $\longrightarrow$ cut I_1 into two halves can construct $I_2 = [a_2, b_2]$ such that $f(a_2) < 0 < f(b_2)$ and length$(I_2) = \frac{1}{2}\text{length}(I_1) = \frac{1}{2^2}\text{length}(I_0)$. Continue inductively to build closed intervals.
   $I_0 \supseteq I_1 \supseteq \ldots \supseteq I_n \supseteq \ldots$ Gives us: $I_n = [a_n, b_n]$ if at any point $f($ midpoint $) = 0$ we have found c. Stop.
   If not we continue, this way we build nested sequence of closed intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq \ldots$ with $I_n = [a_n, b_n]$, $f(a_n) < 0 < f(b_n)$, and length$(I_n) = \frac{1}{2^n}(b-a)_{n\to\infty}0$
   I_n is a nested sequence of closed intervals with length $(I_n) \longrightarrow 0$ by the homeworks we know that $\cap_{m=1}^{\infty} I_n = \{c\}$.
   Claim $f(c) = 0$.

   Proof: $c \in [a_n, b_n], \forall n \longrightarrow a_n \longrightarrow c$ and $b_n \longrightarrow c$, given that $b_n - a_n \longrightarrow 0$. This is from the homework. $a_n \longrightarrow c$ and $b_n \longrightarrow c$ and f is continuous on the closed $[a, b]$. By the sequential criterion for continuity we get that $f(a_n) \longrightarrow f(c)$ and that $f(b_n) \longrightarrow f(c)$.
   By construction $f(a_n) < 0$ and $f(b_n) > 0$. $f(a_n) \longrightarrow f(c)$ and given $f(b_n) \longrightarrow f(c)$, we know by the order limit theorem that $f(c) = \lim f(a_n) \leqslant 0 = \lim f(b_n) \geqslant 0$. For $f(c)$ to satisfy both of these constraints, $f(c) = 0$.

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Example 18.4.1

Question 38: Show that the equation $x\sin(x) + \cos(x)$ has at least one real solution

Solution: Bring x^3 to the left and make the equation on the left a function. Consider $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) = x\sin(x) + \cos(x) - x^3$. Here we emphasize f is continuous because it is comprised of continuous functions.

Compute $f(0) \longrightarrow 1 > 0$, $f(\pi) = -1 - \pi^3 < 0$. f is continuous on the closed interval $[0, \pi]$ and $f(0) > 0$ and $f(\pi) < 0$. By the IVT $\exists c \in (0, \pi)$ such that $f(c) = 0$.

$f(\frac{\pi}{2}) = \frac{\pi}{2} - \frac{\pi^3}{8} < 0$ Then by the IVT $\exists c \in (0, \frac{\pi}{2})$ such that $f(c) = 0$. $f(\frac{\pi}{4}) > 0$ implies that $\exists c \in (\frac{\pi}{4}, \frac{\pi}{2})$ such that $f(c) = 0$.

Note:-

The special case gives us everything.

18.4.3 Corollary

Theorem 18.4.6

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a closed and continuous function. Then $f([a, b]) = [c, d]$. The range of f is another closed interval c, d. The graph has no holes or jumps.

Proof: $[a, b]$ is compact. f continuous by EVT $\exists x_0, x_1 \in [a, b]$ such that $f(x_0) \leqslant f(x) \leqslant f(x_1)$, $\quad \forall x \in [a, b]$ set $c = f(x_0), d = f(x_1)$. We know from $f(x_0) \leqslant f(x) \leqslant f(x_1)$ that $f([a, b]) \subseteq [c, d]$. Then by the IVT if $L \in [f(x_0), f(x_1)] \exists e \in [a, b]$ such that $f(e) = L \longrightarrow [c, d] \subseteq f([a, b])$.

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Chapter 19

The Derivative Chapter 5

Question 39

Given $f : \mathbb{R} \to \mathbb{R}$ and a point $c \in \mathbb{R}$, what does $f'(c)$ represent?
Solution: The derivative represents the slope of the tangent line at $(c, f(c))$.

Definition 19.0.1: Definition of the Derivative

Suppose $f : I \to \mathbb{R}$ function. I = interval, open closed, arbitrary. Let $c \in I$. The derivative of f at c is:

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$
If this limit exists
$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$
If the limit exists.

Definition 19.0.2: Definition of Differentiable

If $f'(c)$ exists we say that f is differentiable at c. We will say that f is differentiable if $f'(c)$ exists for all $c \in I$, the original interval.

Note:-

The $\frac{f(x)-f(c)}{x-c}$ is the slope of the secant line connecting $(x, f(x))$ and $(c, f(c))$. As we allow $x \to c$, the secant line becomes tangent.

Example 19.0.1

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. And $c \in \mathbb{R}$. Compute $f'(c)$. We should expect $2c$.

For $x \neq c$ we have:

$$\frac{f(x)-f(c)}{x-c} = \frac{x^2-c^2}{x-c}$$
$$\frac{(x-c)(x+c)}{x-c} = x + c.$$
This gives:
$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c}(x + c) = c + c = 2c$$
. Note that $x + c$ is a polynomial function continuous at c.

Example 19.0.2 (Prove the Power Rule)

$f : \mathbb{R} \to \mathbb{R}, c \in \mathbb{R}$. We want to show that $f'(c) = nc^{n-1}$. Let $x \neq c$. For $n \geqslant 0$ and $n \in \mathbb{Z}$. Then:

$$\frac{f(x)-f(c)}{x-c} = \frac{x^n-c^n}{x-c}.$$

Recall $x^n - c^n = (x-c)(x^{n-1} + cx^{n-2} + \ldots + xc^{n-2} + c^{n-1})$
$\frac{x^n - c^n}{x-c} = x^{n-1} + cx^{n-2} + \ldots + c^{n-1}x + c^{n-1} = p(x)$
$\lim_{x\to c} \frac{x^n-c^n}{x-c} = \lim_{x\to c} p(x) = p(c)$.
If we plug $x = c$ into $p(c) = \underbrace{c^{n-1} + c^{n-1} + \ldots + c^{n-1}}_{n \text{ times}} = nc^{n-1}$.

Example 19.0.3

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$. $f'(0)$ should not be differentiable. The picture suggests that you can create at least two tangent lines.

At the end of the day the two sided limits will be different. Enough to show (practice problems) that $\lim_{x\to 0^-} \frac{f(x)-f(c)}{x-0}$ and $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0}$ are not equal.

If $x \in \mathbb{R}$, $x < 0$ then $\frac{f(x)-f(0)}{x-0} = -1$ if $y \in \mathbb{R}, y > 0$ then $\frac{f(y)-f(c)}{y-0} = 1$.

$f(x) = |x|$:

$$f'(x) = \begin{cases} 1, x > 0 \\ -1 x < 0 \\ \text{DNE}, x = 0 \end{cases}$$

19.1 Thursday November 14th

Note:-

1. Two Homeworks Left, one on differentiablity one on integration.
2. The homework on differentiablity will be due the tuesday before Thanksgiving.
3. The Last Homework, Homework 11 will be due during the reading period.

19.1.1 Reminders

1. $f : I \to \mathbb{R}$ I = interval. $c \in I$, this implies that c is a limit point of I. Then we say that f is differentiable at c if $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists.
2. Last time we saw some examples. We did the power rule:
   $f(x) = x^n$ where $n \in \mathbb{N}$ then $f'(x) = nx^{n-1}$.
3. We also showed an example $f(x) = |x|$ is not defined at $c = 0$

   Example 2 implies that f continuous at a point does not imply that f is differentiable at c.

Note:-

Let $f : A \to \mathbb{R}$ and $c \in A$. c is either a limit point or an isolated point.

1. If c is isolated, by the homework and the review, we know that f is always continuous at this point.
2. If c is a limit point of A, then f is continuous at $c \leftrightarrow \lim_{x\to c} f(x) = f(c)$.

Theorem 19.1.1 Differentiability implies continuity

If f is differentiable at c then it is continuous at c.

Proof: We define the derivative only for limit points, at interior points of intervals, at points where it makes sense to take the limit.

We want to show that f is continuous at c. It is enough to show $\lim_{x\to c} f(x) = f(c)$.

Or equivalently that $\lim_{x\to c}(f(x) - f(c)) = 0$. By algebra of limits. This is what we will prove.

We know $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists.

We also know $\lim_{x\to c}(x - c)$, also exists and also it is equal to zero. This is because $(x - c)$ is a polynomial. The limit of a polynomial is equal to $p(x)$ which is equal to zero in this case.

Now we proceed with algebra of limits. $\lim_{x\to c}(f(x) - f(c)) = \lim_{x\to c}(\frac{f(x)-f(c)}{x-c}(x - c))$. And now because each of the limits exist:

$\lim_{x\to c} \frac{f(x)-f(c)}{x-c)=f'(c)}$ and $\lim_{x\to c}(x - c) = 0$ both exist by algebra of limits we get:

$$\lim_{x\to c}(f(x) - f(c)) = (\lim_{x\to c} \frac{f(x)-f(c)}{x-c}) \lim_{x\to c}(x - c).$$

This is equal to $f'(c) \cdot 0 = 0$.

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Theorem 19.1.2 Algebra of differentiable functions

Suppose we are given $f, g : I \to \mathbb{R}, \quad c \in I$. Suppose f, g are differentable at c.

Then

1. $f + g$ is differentiable at c and $(f + g)'(c) = f'(c) + g'(c)$
2. *Product Rule fg is differentiable at c and $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$
3. (Quotient Rule) assume $g(c) \neq 0$. Then $\frac{f(x)}{g(x)}$ is differentiable at c.

   The formula is:

   $\frac{f(x)}{g(x)}'(c) = \frac{f'(c)g(c)-f(c)g'(c)}{g(c)^2}$

Proof: Proof of 1 above. Let $x \in I$ and $x \neq c$.

$$\frac{f(x)+g(x)-f(c)-g(c)}{x-c} =$$
$$\frac{(f(x)-f(c))+(g(x)-g(c))}{x-c}$$
$$\frac{f(x)-f(c)}{x-c} + \frac{g(x)-g(c)}{x-c}$$

We know that $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = f'(c)$ exists and we also know that $\lim_{x\to c} \frac{g(x)-g(c)}{x-c} = g'(c)$ exists.

These two facts, and using algebra of limits we know that the following limit exists:

$$\lim_{x\to c} \frac{f(x)-f(c)+g(x)-g(c)}{x-c}$$
$$= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} + \frac{g(x)-g(c)}{x-c}$$
$$= f'(c) + g'(c)$$

Note:-
This concludes one. Nothing too hard, just make sure you don't split limits if they don't necessarily exist.

Note:-
Proof of 2 below

Let $x \in I$ and $x \neq c$. Then:

$$\frac{f(x)g(x)-f(c)g(c)}{x-c}$$

We have seen this trick before we will add and subtract as below:

$$\frac{f(x)g(x)-f(x)g(c)+f(x)g(c)-f(c)g(c)}{x-c}$$
$$= \frac{f(x)g(x)-f(x)g(c)}{x-c} + \frac{f(x)g(c)-f(c)g(c)}{x-c}$$
$$f(x)\frac{g(x)-g(c)}{x-c} + g(c)\frac{f(x)-f(c)}{x-c}$$

Now we can proceed as we did before. We know on the left hand side that $\lim_{x\to c} \frac{g(x)-g(c)}{x-c} = g'(c)$ and we also

know that $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = f'(c)$.
We also know have two more guys to worry about. $f(x)$ and $g(c)$.
f is differntiable at $c \longrightarrow f$ is continuous at $c \longrightarrow \lim_{x\to c} f(x) = f(c)$.
We don't have to worry about $g(c)$ because it is a constant.
Then by Algebra of limits we get:

$$\lim_{x\to c} \frac{f(x)g(x)-f(c)g(c)}{x-c} =$$
$$\lim_{x\to c} f(x) \lim_{x\to c} \frac{g(x)-g(c)}{x-c} + g(c) \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$
$$= f(c)g'(c) + f'(c)g(c)$$

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Note:-

Chain Rule. 3 Above will be left for Homework. Not going to be proven because of time. Very likely the TA, Aaratrick will be asked to prove next Wednesday.

Theorem 19.1.3 Chain Rule

Suppose $f : A \longrightarrow \mathbb{R}$ and $g : B \longrightarrow \mathbb{R}$ functions such that $f(A) \subseteq B \leftrightarrow g \circ f$ is defined.

Assume f is differentiable at c and g is differentiable at $f(c)$.

Then $g \circ f$ is differentiable at c and $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.

Example 19.1.1

1. $f_0(x) = \sin(\frac{1}{x}), x \neq 0$.

   We showed that $\lim_{x\to 0} f(x)$ Then there is no way that we can define $f(x)$ at $x = 0$ to make $f(x)$ continuous.

   We say that f has an essential discontinuity at 0.

2. $g(x) = \begin{cases} x\sin(\frac{1}{x}), x \neq 0 \\ 0, x = 0 \end{cases}$

   BUT, $g(x)$ is not differentiable at $c = 0$. $\lim_{x\to 0} \frac{g(x)-g(0)}{x-0} = \lim_{x\to 0} \sin(\frac{1}{x})$, again which does not exist.

3. $h(x) = \begin{cases} x^2\sin(\frac{1}{x}), x \neq 0 \\ 0, x = 0 \end{cases}$

   This gives us:

   $\lim_{x\to 0} \frac{h(x)-h(0)}{x-0} = \lim_{x\to 0}(x \sin \frac{1}{x} = 0$.

Note:-

Check $h'(x)$ is not continuous at $x = 0$.

Note:-

The main takeaway is that we shouldn't make any assumptions. Continuous does not imply differentiable. Differentiable does not imply that the derivative is continuous. All we know is that if f is differentiable everywhere then f is continuous.

If $f : (a, b) \longrightarrow \mathbb{R}$ is differentiable, this does NOT imply if f' is continous on (a, b).

Note:-

$\exists f \mathbb{R} \to \mathbb{R}$ which is continuous everywhere, differentiable nowhere!

19.2 Big Theorems for Differntiablity

Theorem 19.2.1 (Interior Extrema Theorem)

Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. Suppose f attains a maximum or a minimum value at some point $c \in (a, b)$. Then $f'(c) = 0$.

Proof: The two case of max and minimum are the same. We will prove the maximum. Suppose f attains a maximum at c.

This implies that $f(x) \leqslant f(c) \quad \forall x \in (a, b)$.

$c \in (a, b)$, Since c is an intereior point of (a, b) we can find 2 sequences a_n and b_n such that $a_n, b_n \in (a, b), \quad \forall n \geqslant 1$ and $a_n, b_n \to c$. Where $a_n < c < b_n$ for all n.

We also know that f is differentiable at c. This means that $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = f'(c)$ exists.

By the sequential criterion for functional limts, we get the following:

$$a_n \to c \to \frac{f(a_n)-f(c)}{a_n-c} \to f'(c)$$

Similarly, $b_n \to c \to \frac{f(b_n)-f(c)}{b_n-c} \to f'(c)$.

We have $\frac{f(a_n)-f(c)}{a_n-c}$ because $f(c)$ is a maxmimum we know that the numerator is smaller than or equal to 0.

Because $a_n - c$ approaches c from the left we get that the fraction is $\geqslant 0$.

This implies that $f'(c) \geqslant 0$. For the b_n's we get the other way around implying that $f'(c) = 0$.

$\frac{f(b_n)-f(c)}{b_n-c}$ The numerator is again $\leqslant 0$ but the denominator is positive becuase we are approaching c from b_n.

This implies that:

$f'(c) \leqslant 0$. The only option for both of these limits to be true is for $f'(c) = 0$.

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19.2.1 Sequential Criterion for functional limits

1. Our function $f : A \to \mathbb{R}$ and c is a limit point. $\lim_{x\to c} f(x) = L \leftrightarrow$ for every sequence $a_n \subseteq A$ with $a_n \to c$ and $a_n \neq 0$ it follows that $f(a_n) \to L$
2. In the proof we applied the forward direction for the function $g(x) = \frac{f(x)-f(c)}{x-c}$ Note that c is not in the domain.

19.2.2 Remarks on the Theorem

1. The converse is not true. Caution. Take $f(x) = x^3$. $f : \mathbb{R} \to \mathbb{R}$. $f'(0) =$ but we do not have a maximum or minimum value there. $f'(x) = 3x^2$ but $f'(0) = 0$ and the funciton clearly has greater values than 0.
2. Interior Extrema Theorem Still Holds for $f(c)$ local maximum or minimum.

   Definition 19.2.1: Local Maximum or Minimum

   Let $f : (a, b) \to \mathbb{R}$. We say that f attains a local maximum at c if $\exists (d, e) \subsetneq (a, b)$ with $c \in (d, e)$ and $f(x) \leqslant f(c)$ for all $x \in (d, e)$. Similarly for local minima.
3. Interior Extrema Theorem can be modified to show: If f attains a local max or min at c and f is differentiable at c then $f'(c) = 0$.
4. Suppose $f : (a, b) \to \mathbb{R}$ continuous. Suppose f attains a local extreme value at $c \in (a, b)$. Then $f'(c) = 0$ or the derivative does not exist. Absolute value or square root, fall into the second category. $f(x) = |x|$ and $g(x) = \sqrt{|x|}$ in the case of the pigeon the derivative goes to infinity. In both cases $f(0)$ is a min but $f'(0)$ does not exist.

19.2.3 Closed Intervals Example

Example 19.2.1

Suppose $f : [a, b] \to \mathbb{R}$ which is continuous. a continuous funciton on a closed interval. By the Extreme Value theorem, we know that f attains a maxium and a minimum value.

The max and the min can happen at the following places:

1. $c \in (a, b)$ where $f'(c) = 0$
2. $c \in (a, b)$ where $f'(c)$ does not exist or
3. $c = a$ or $c = b$.

Theorem 19.2.2 (Darbaux's Theorem)

Suppose $f : [a, b] \to \mathbb{R}$ and it is differentiable on this interval. Then f' satisfies the Intermediate Value Theorem.

It will satisfy IVT even when it is not continuous. (it obviously satisfies IVT if it is continuous)

I.e. if α is # between $f'(a)$ $f'(b)$, then $\exists c \in [a, b]$ such that $f'(c) = \alpha$.

19.3 Thursday November 21st

Note:-

No office Hours Today. Office Hours on Monday. Similar to Last Week. 1:30 - 3:00 PM.

19.3.1 Reminers

1. Interior Extreme Theorem:
   Let $f : I \to \mathbb{R}$ on an interval I. And we assume that f is differentiable. Suppose that f attains a local max or minimum value at some point $c \in I$. Then we conclude that $f'(c) = 0$.

Note:-

Three Theorems on Derivatives

Theorem 19.3.1 Darbaux's Theorem

Suppose $f : [a, b] \to \mathbb{R}$ which is differentiable. Let α be an intermediate value between $f'(a)$ and $f'(b)$. Then $\exists c \in [a, b]$ such that $f'(c) = \alpha$. In other words, the function $f' : [a, b] \to \mathbb{R}$ satisfies the intermediate value theorem.

Note:-

Note that if f' is continuous on the closed $[a, b]$ then, this is regular IVT. All that we know is that f is continuous and this has no implications on whether f' is continuous.

Proof: The proof has two main steps. One is left for Homework. For regular IVT we reduced for the special case. We do so similarly here, we will reduce to a special case.

Step 1. Reduce to the special case when $\alpha = 0$. It is not required that $\alpha = 0$. So we will need to subtract. Consider a new function $g : [a, b] \to \mathbb{R}$ with $g(x) = f(x) - \alpha(x)$. It is clear that g is differentiable on the closed $[a, b]$ because both $f(x)$ and $\alpha(x)$ are. We have shown that $g'(x) = f'(x) - \alpha$. By assumption α is a number between $f'(a)$ and $f'(b)$. Without loss of generality assume that $f'(a) \leqslant \alpha \leqslant f'(b)$. Because of this $g'(a) \leqslant 0$ and $g'(b) \geqslant 0$. Thus, 0 is an intermediate value for g'. If we prove $\exists c \in [a, b]$ such that $g'(c) = 0$. Then we are done becuase this implies that $f'(c) = \alpha$.

Step 2. Will be to prove Darboux when $\alpha = 0$. This is HW 10.

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Theorem 19.3.2 Mean Value Theorem

Let $f : [a, b] \to \mathbb{R}$. Suppose f is continuous on the closed interval $[a, b]$ and differentiable on the open. Then we can conclude $\exists c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(c)}{b-a}$.

Special Case: Rolle's Theorem:

Let $f : [a, b] \to \mathbb{R}$ Assume f is continuous on the closed, f is differentiable on the open, and $f(a) = f(b)$ then the conclusion is that $\exists c \in (a, b)$ such that $f'(c) = 0$.

19.3.2 Geometric Interpretation of Rolle

$\exists c \in (a, b)$ where the tangent line at $(c, f(c))$ is horizontal. This is under the assumption that there exists $f(a) = f(b)$. Without this assumption the MVT says: $\exists c \in (a, b)$ such that the tangent line at $(c, f(c))$ is parallel to the secant line from AB.

Question 40: True False for MVT

1. $f : [a, b] \to \mathbb{R}$ such that f is differentiable on $[a, b] \to \exists c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.
2. $f : [a, b] \to \mathbb{R}$ suppose f is differentiable on (a, b) with $f(a) = f(b)$. Then $\exists c \in (a, b)$ such that $f'(c) = 0$.

Solution: True. The assumption is stronger. If the assumption is differentiable on the closed interval then it is continuous on the closed and differentiable on the open. ***Solution:*** False. Take a linear function that has jump discontinuity at the end points such that $f(a) = f(b)$.

Proof: Proof for Rolle. Pay Attention because you will do something similar in the homework. The main idea is that you have a continuous function on a closed interval, so we can apply the EVT.
f attains a maximum and a minimum value on the closed interval.
By the interior extremum theorem $\to$ the max and the min can only happen at points $c \in (a, b)$ where $f'(c) = 0$ or at the endpoints.
Case 1: Both the max and the min occur at endpoints. Then Since $f(a) = f(b)$, this implies that max = min and this can only happen if the function is constant throughout $[a, b]$. If the function is constant then the derivative is zero everywhere and we are done.
Case 2: Suppose at least one of the max | min occurs at an interior point, $c \in (a, b)$. Then by th eInterior Extreme Theorem we know that $f'(c) = 0$. for some c.

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19.3.3 Proof for MVT

Proof: Consider a new function $g : [a, b] \to \mathbb{R}$ with $g(x) =$ secant line:

$$g(x) = \left(\frac{f(b)-f(a)}{b-a}\right)(x-a) + f(a)$$

Now take $h(x) = f(x) - g(x)$. For $x \in [a, b]$. Because g, f are closed and differentiable we know that h is closed and differentiable as well. The nice thing is that $h(a) = h(b)$ on the endpoints. In fact $h(a) = h(b) = 0$. This implies that Rolle's Theorem applies for $h(x) \to \exists c \in (a, b)$ such that $h'(c) = 0$.

$$h'(x) = f'(x) - g'(x) =$$
$$= f'(x) - \frac{f(b)-f(a)}{b-a}$$
$$\to \exists c \in (a, b) \text{ such that } f'(c) = \frac{f(b)-f(a)}{b-a}.$$

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19.3.4 Applications

Example 19.3.1

Let $f : I \to \mathbb{R}$ differentiable. Where I is an interval. Suppose that $f'(x) = 0$ everywhere. Then $f(x) =$ constant $= k$.

Proof: Suppose for contradiction that f is not constant. This implies $\exists x < y$ with $x, y \in I$ such that $f(x) \neq f(y)$. Now, f is differentiable on I and $[x, y] \subseteq I \to f$ is differentiable on the closed $[x, y]$. Now we are in the situation of the first true false above. We have stronger assumptions than needed. By the MVT $\exists c \in (x, y) \subseteq I$ such that $f'(c) = 0 = \frac{f(y)-f(x)}{y-x} \neq= 0$. Which is a contradiction.

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Example 19.3.2

Let $f : I \to \mathbb{R}$ differentiable suppose $f'(x) > 0, \forall x \in I \to f$ is strictly increasing on I. Meaning if $x < y \to f(x) < f(y)$.

Proof: Let $x, y \in I$ with $x < y$ apply MVT on $[x, y]$. $\exists c \in (x, y)$ with $f'(c) = \frac{f(y)-f(x)}{y-x}$. Because $f'(c) > 0$ we know that $f(y) - f(x) > 0$ which creates a contradiction.

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Partial Converse

$f : I \to \mathbb{R}$ differentiable. If f is strictly increasing then $f'(x) \geqslant 0$ for all $x \in I$. Consider $f(x) = x^3$. This function is strictly increasing but $f'(0) = 0$.

Question 41: T-F

Suppose $f : A \to \mathbb{R}$ differentiable with A open. Suppose $f'(x) = 0$ for all x. Then $f(x) = k$ constant. Take $f : (1, 2) \cup (3, 4) \to \mathbb{R}$.

19.4 Riemann Integrals

Suppose we have a function $f : [a, b] \to \mathbb{R}$, $f(x) > 0$. What does $\int_a^b f(t)dt$ represent and how do we compute it? We expect the integral to be the area under the graph. Define an anti derivative $F(x)$ meaning a function that satisfies $F'(x) = f(x)$. And then we compute the integral of little $f(x) = F(b) - F(a)$. Two immediate goals for us will be to prove these things.

Example 19.4.1

Take $f : [1, 4] \to \mathbb{R}$ where $f(x) = \lfloor x \rfloor$ Pictorally we would expect the integral to exist. But we claim that we can never find an anti derivative. Suppose for contradiction that such a capital F existed. Then apply Darbaux. The derivatives function must satify IVT as a result. Because f doesn't satisfy IVT then it is impossible that F exists.

19.4.1 Setup

Begin with $f : [a, b] \to \mathbb{R}$ and the only assumption that I will make is f is bounded. This means $\exists m, M \in \mathbb{R}$ such that $m \leqslant f(x) \leqslant M$.

Definition 19.4.1: Definition of Partition

A partition of the interval $[a, b]$ is a division of $[a, b]$ into a finite number of closed sub intervals.

The notation for these partitions will be $P = \{a = x_0 < x_1 < x_2 < \ldots < x_n = b\}$. For a is the partition of $[a, b]$ into n sub intervals. Want n subintervals which is why we label the first point $a = x_0$.

For $k = 1, \ldots, n$ set $I_k = [x_{k-1}, x_k]$. f is bounded I_k which means that $m_k = \inf\{f(x) : x \in I_k\}$ and capital $M_k = \sup\{f(x) : x \in I_k\}$

19.5 Tuesday November 26st

Note:-

This is an important lecture. Starting new stuff, and so must keep going. We have missed two classes this semester.

Note:-

This test was definitely harder. The final will be only a little bit longer, and you will have 3 hours to prepare for it. Getting a 50 on this test was still a very decent grade. The final will be somewhere in the middle between this test and the last test.

19.6 Setup

1. $f : [a, b] \to \mathbb{R}$ are only assumption right now is that f is bounded. This means that $M = \sup\{f(x) : x \in [a, b]\}$ and $m = \inf\{f(x) : x \in [a, b]\}$.
2. We separate the interval $[a, b]$ into a number of closed sub intervals. $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$. These sub intervals do not need to be of equal length.
3. $I_i = [x_{i-1}, x_i]$ for $i = 1, \ldots, n$. This gives us n sub intervals of $[a, b]$. For each i let $m_i = \inf\{f(x) : x \in I_i\}$ and $M_i = \sup\{f(x) : x \in I_i\}$.
4. We form the upper sum of f with respect to P as: $U(f; P) = \Sigma_{k=1}^{n} M_k(x_k - x_{k-1})$
5. And the lower sum of f with respect to P as: $L(f; P) = \Sigma_{k=1}^{n} m_k(x_k, x_{k-1})$

Definition 19.6.1: Upper and Lower Sum

See above

Note:-

The upper and lower sum is a left and right rieman sum. (Observation from a graph drawn in class)

Note:-

Intuitively if we make more subintervals we should expect our respective estimations to grow closer to one another converging on the actual area under the curve. Note that these sums over estimate and under estimate depending on the function.

Definition 19.6.2: Partition

A partition Q of $[a, b]$ is a refinement of P if $Q \subseteq P$.

Meaning $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ and Q adds more points. Q definitely contains the $x_{i's}$ and possibly more.

Theorem 19.6.1 Lemma 1

Suppose P, Q are partitions of $[a, b]$ with $P \subseteq Q$ meaning Q is a refinement of Q. Then $L(f; P) \leqslant L(f; Q)$ and $U(f; Q) \leqslant U(f; P)$. This implies that $U(f, Q) - L(f; Q) \leqslant U(f; P) - L(f; P)$. And this is where we are going.

Proof: $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ and $m_k = \inf\{f(x) : x \in I_k\}$ and $M_k = \sup\{f(x) : x \in I_k\}$. We will focus on the interval $I_k = [x_{k-1}, x_k]$. We know that $P = \{a < \ldots < x_{k-1} < x_k < \ldots < x_n = b\}$.

1. Case 1. Q does not contain any new points between x_{k-1}, x_k in this case the summands $m_k(x_k - x_{k-1})$ and $M_k(x_k - x_{k-1})$ in $L(f; Q)$ and $U(f; Q)$ stay the same. This means we did not add any new points in our subinterval.

2. Case 2. Suppose Q contains new points. Suppose in Q we have $Q = \{a < \ldots < x_{x-1} < z_1 < z_2 < \ldots < z_m < x_k < \ldots < x_n = b\}$. We will do induction.
   Induction on $m \geqslant 1$. Base Case: $m = 1$. $x_{k-1} < z_1 < x_k$. This implies that I_k is split into two sub intervals. $I^1_{k_1} = [x_{k-1}, z_1]$ and $I^2_k = [z_1, x_k]$. Additionally we have $m^1_k = \inf\{f(x) : x \in I^1_k\}$ and $M^1_k = \sup\{f(x) : x \in I^1_k\}$, similarly for m^2, M^2. $m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$, if $x \in I^1_k \to x \in I_k \to f(x) \leqslant m_k, \forall x \in I^1_k$. This is because m_k is a lower bound of I_k and hence also of I^1_k. $m^k_1 =$ the largest lower bound of I^1_k and which implies that $m^1_k \geqslant m_k$ and $m^2_k \geqslant m_k$ similarly, M_k is an upper bound of I_k and $I^1_k \subseteq I_k \to M_k$ is an upper bound of I^1_k, and M^1_k is the least upper bound of I^1_k implies that $M^1_k \leqslant M_k$ and $M^2_k \leqslant M_k$. The same is true for our M^2 sub interval. In $L(f; P)$ we have a summand $m_k(x_k - x_{k-1}) = m_k(x_k - z_1 + z_1 - x_{k-1}) = m_k(x_k - z) + m_k(z_1 - x_{k-1})$. And now we use the relations defined above for $m^1_k \geqslant m_k$ and $m^2_k \geqslant m_k$ to say that:
   $m_k(x_k - z_1) + m_k(z_1 - x_{k-1}) \leqslant m^2_k(x_k - z_1) + m^1_k(z_1 - x_{k-1})$, and we know that this appears in $L(f; Q)$. This argument is identical for $U(f; Q)$ except $U(f; Q) \leqslant U(f; P)$. Induction step is set as exercise.

Note:-

One of the two infimums or supremums will be equal to the original.

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Theorem 19.6.2 Lemma 2

Let P_1 and P_2 be any 2 partitions of $[a, b]$. Then no matter what happens $L(f; P_1) \leqslant U(f; P_2)$. No matter what P_1, P_2 are.

Proof: Consider the partition $Q = P_1 \cup P_2$. Then Q is a common refinement. Now when we take $L(f; P_1)$ lemma 1 gives me that $L(f; Q)$ and that $\leqslant U(f; Q)$ because U is defined with supremums and L is defined with supremums.

$U(f; Q) \leqslant U(f; P_2)$ by Lemma 1, and these two facts imply that $L(f; P_1) \leqslant U(f; P_2)$.

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19.6.1 Observations

1. Let $A_f = \{L(f; P) : P$ partion of $[a, b]]\}$ to be the set of all lower sums, lower partitions. Additionally, we take $B_f = \{U(f; P) : P$ partition of $[a, b]\}$. Fix a partition Q of $[a, b]$. Then by Lemma 2 $\to L(f; P) \leqslant U(f; Q) \forall P$ partitions of $[a, b]$. Then by Lemma 2 again, $U(f; Q)$ is an upper bound of of A_f and $L(f; Q)$ is a lower bound of B_f. The axiom of completeness impliues that $U(f) = \inf\{U(f; P) : P$ partition of $[a, b]\}$ exists, similarly $L(f) = \sup\{L(f; P) : P$ partition of $[a, b]\}$ exists.

Definition 19.6.3: Integrable

We say that $f : [a, b] \to \mathbb{R}$ if $U(f) = L(f) = \int_a^b f(x)dx$. Sometimes $L(f)$ is called the left integral and $U(f)$ is the right integral. Can be thought of as L and R limit. This is what we call the Riemann Integrable.

19.6.2 Things that we want to prove

Theorem 19.6.3 Epsilon Criterion for Integrability

$f:[a,b]\to\mathbb{R}$, bounded. Then f is integrable $\leftrightarrow \forall\epsilon>0\exists\ P_\epsilon$ of $[a,b]$ such that $U(f;P_\epsilon)-L(f;P_\epsilon)<\epsilon$.

Proof: $(\leftarrow)$. Suppose $\forall\epsilon>0\exists P_\epsilon$ partition such that $U(f;P_\epsilon)-L(f;P_\epsilon)<\epsilon$. We want to show that $U(f)=L(f)$.

Note:-

1. We know that $L(f)\leqslant U(f)\to U(f)-L(f)\geqslant 0$.
2. It suffices to show that $\forall\epsilon>0 \quad U(f)<L(f)<\epsilon$. This will be now our goal

Let $\epsilon>0$. $\to\exists P_\epsilon$ partition of $[a,b]$ such that $U(f;P_\epsilon)-L(f;P_\epsilon)<\epsilon$.
$U(f)=\inf\{U(f;P):P \text{ is a partition}\}$ This implies that $U(f)\leqslant U(f;P_\epsilon)$ this implies that $L(f)=\sup\{L(f;P):P \text{ part of } [a,b]\}\to L(f)\geqslant L(f;P_\epsilon)$. These two facts together implies that $U(f)-L(f)\leqslant U(f;P_\epsilon)-L(f;P_\epsilon)<\epsilon$.
$(\to)$ Suppose now that f is integrable on $[a,b]$. By definition this means $U(f)=L(f)$.
We want to show that $\forall\epsilon>0\exists P_\epsilon$ partition such that $U(f;P_\epsilon)-L(f;P_\epsilon)<\epsilon$.
Let $\epsilon>0$. $U(f)=\inf\{U(f;P):P \text{ is a partition of}[a,b]\}\to U(f)+\frac{\epsilon}{2}$ is not a lower bound of the set. Therefore $\exists P_1$ partition of $[a,b]$ such that $U(f;P_1)<U(f)+\frac{\epsilon}{2}$.
$L(f)=\sup\{L(f;P):P \text{ is a partition of } [a,b]\}$. This implies that $L(f)-\frac{\epsilon}{2}$ is not an upper bound, which implies that I can find another partition of $[a,b]$ such that $L(f;P_2)>L(f)-\frac{\epsilon}{2}$. Take $P_\epsilon=P_1\cup P_2$, $L(f;P_\epsilon)\geqslant L(f_1;P_2)>L(f)-\frac{\epsilon}{2}$, and $U(f;P_\epsilon)\leqslant U(f;P_1)<U(f)+\frac{\epsilon}{2}$. This implies that $U(f;P_\epsilon)-L(f;P_\epsilon)<\epsilon$.
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Note:-

Continuous implies integrable. Proof is omitted for time.

Example 19.6.1

$f:[0,2]\to\mathbb{R}$ for $f(x)=\begin{cases}1, & x\neq 1\\ 0, & x=1\end{cases}$ My function is not continuous, but it has an area under the curve.

We expect $\int_0^2 f=2$.

Note:-

If $P=\{0<x_0<x_1<\ldots<x_n=2\}$ is an arbitrary partition of $[0,2]$ then $U(f;P)=2$. This is because in every subinterval $[x_{k-1},x_k]$, $M_k=\sup\{f(x):x\in I_k\}=1$. This gives us $U(f;P)=\Sigma_{k=1}^n M_k(x_k-x_{k-1})=\Sigma_{k=1}^n(x_k-x_{k-1})=x_n-x_0=2-0$.

Let $\epsilon>0$. We look for $\delta>0$ such that for the partition $P=\{0<1-\delta<1+\delta<2\}$ such that $U(f;P)-L(f;P)<\epsilon$. We commute and then we force δ to guarantee this.

Note:-

$I_1=[0,1-\delta]$, and $\delta x_1=x_1-x_0=1-\delta$. $I_2=[1-\delta,1+\delta]$ and $\delta x_2=1+\delta-1+\delta=2\delta$. $I_3=[1+\delta,2]$, $\delta x_3=2-1-\delta=1-\delta$. We have $M_1,M_2,M_3=1$ everywhere else $m_1=m_3=0$ while m_2 is our hole.

$U(f;P)-L(f;P)=(M_1-m_1)\delta x_+(M-2-m_2)\delta x_2+(M_3-m_3)\delta x_3$. 2 of our terms go away which gives us $\delta x_2=2\delta$. This is enough to choose $0<\delta<\frac{\epsilon}{2}$.
We can conclude that f is indeed integrable on $[0,2]$.

19.7 Tuesday December 03

1. Last HW is up

2. Pickup today where we left off yesterday, today is a very important lecture.

3. Quiz Tomorrow, on the integration problem

19.7.1 Reminders

1. $f : [a, b] \to \mathbb{R}$ bounded, f is integrable if and only if $U(f) = L(f) = \int_a^b f$ where $U(f) = \inf\{U(f; p)\}$: the infimum of the upper sums, and p is a partition of $[a, b]$. $L(f) = \sup\{L(f; p)\}$ is the infimum of the lower sums

2. $L(f; P) \leqslant U(f; Q) \forall p, q$ this implies that $L(f) \leqslant U(f)$.

3. KEY: ϵ criterion for integrability:
   f is integrable on $[a, b] \leftrightarrow \forall \epsilon > 0 \exists P$ part of $[a, b]$ such that $U(f; p) - L(f; p) < \epsilon$

Note:-

Today we show continuous functions are integrable, and that this fact is not necessary; we can integrate discontinuous functions.

Theorem 19.7.1 1

Let $f : [a, b] \to \mathbb{R}$ continuous on $[a, b]$. Then f is integrable.

Note:-

Preparation for this proof

As a reminder, f continuous at the point c means $\forall \epsilon > 0 \exists \delta > 0$ such that if $|x - c| < \delta \to |f(x) - f(c)| < \epsilon$. In general: given $\epsilon > 0$, we find a $\delta > 0$ which depends on ϵ and on the point c. It would be nice if we can find a uniform delta that doesn't have dependence on all of the c.

Definition 19.7.1

$f : A \to \mathbb{R}$ we say that f is uniformly continuous on A if:
$\forall \epsilon > 0 \exists \delta > 0$ depending only on ϵ such that for every $x, y \in A$ with $|x - y| < \delta \to |f(x) - f(y)| < \epsilon$.

Note:-

With the exception of constant functions, the δ always depends on $\epsilon > 0$.

Note:-

In the review sheet you will see $f(x) = x^2, f : \mathbb{R} \to \mathbb{R}$ is not uniformly continuous.

What you will prove in the review is that if $f : [a, b] \to \mathbb{R}$ is continuous on a closed interval then it is uniformly continuous.

Note:-

$P = \{a = x_0 < x_1 < \dots < x_n = b\}$ a partition, we note that $\Delta x_k = x_k - x_{k-1}$ for $k = 1, \dots, n$

We also say that P is of equal length if Δx_k =constant.

For $P = \{a = x_0 < x_1 < \dots < x_n = b\}$ we have for equal length $\Delta x = \frac{b-a}{n}$

Proof: We want to show that $\forall \epsilon > 0 \exists P$ part of $[a, b]$ such that $U(f; P) - L(f; P) < \epsilon$.
Let $\epsilon > 0$. We will use uniform continuity of the funciton. Since f is uniformly continuous on the closed $[a, b]$.
For $\epsilon' = \frac{\epsilon}{b-a} \exists \delta > 0$ such that if $x, y \in [a, b]$ with $|x - y| < \delta$ then $|f(x) - f(y)| < \frac{\epsilon}{b-a}$.
Choose a partition partition p of $[a, b]$ of equal length. $\Delta x < \delta$. We can find such a partition as a result of the Archimedian Property. $\exists n \in \mathbb{N}$ such that $\frac{b-a}{n} < \delta$.

On each subinterval, $[x_{k-1}, x_k], f$ is continuous $\longrightarrow$ by EVT, f attains a maximum and minimum value. This means that $\exists y_k, z_k \in [x_{k-1}, x_k]$ such that $f(y_k) \leqslant f(x) \leqslant f(z_k)$ for all $x \in [x_{k-1}, x_k]$. This gives me now in the interval that $m_k = f(y_k)$ and the capital $M_k = f(z_k)$.

$U(f;P) - L(f;P) = \Sigma_{k=1}^{n}(M_k - m_k)(x_k - x_{k-1}) = \Sigma(f(z_k) - f(y_k))\Delta x$

Since $y_k, z_k \in [x_{k-1}, x_k]$ and $\Delta x < delta$ then $|y_k - z_k| < \delta$. By the uniform continuity that we have above, this gives me $|f(y_k) - f(z_k)| < \frac{\epsilon}{b-a} \longrightarrow f(z_k) - f(y_k) < \frac{\epsilon}{b-a}$

Now $U(f;P) - L(f;P) = \Sigma_{k=1}^{n}(f(z_k) - f(y_k))\Delta x$. Each one of these guys is $< \frac{\epsilon}{b-a}\Sigma_{k=1}^{n}\Delta x = \frac{\epsilon}{b-a}(b-a) = \epsilon$

Note that the sum of the Δx is the length of the interval, it is a telescoping series.

In the hw you will prove that if $f : [a, b] \longrightarrow \mathbb{R}$ is increasing then f is integrable, without assuming that f is continuous. Take equal lengths and try to adjust.

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Example 19.7.1 (Discontinuities)

Take $f : [-1, 1]$ to be the following function:

$$f(x) = -1 \text{ when } x < 0 \text{ and } 2, x \in [0, 1) \text{ and } 1 \text{ when } x = 1$$

Given $\epsilon > 0$ we want to find partition of $[-1, 1]$ such that $U(f;P) - L(f;P) < \epsilon$. Look for partition of the form $P = \{-1 < -\delta < \delta < 1 - \delta < 1\}$, meaning the least possible partitions show that I circle around the points of discontinuity. This creates for intervals $[-1, -\delta], [-\delta, \delta], [\delta, 1 - \delta], [1 - \delta, 1]$.

Compute all of the m_i, M_i and all of the Δx for each of these intervals.

Look at the value of the function for the supremums and the infimums. Then we calculate the Δx to by subtracting $b - a$ to get the δx.

This gives us enough information to compute $U(f;P) - L(f;P) = 7\delta$ enough to take $\delta < \frac{\epsilon}{7}$

Guess if $f : [a, b] \longrightarrow \mathbb{R}$ is bounded and has only finitely many discontinuities $\longrightarrow f$ is integrable. This is true, pretty much the example is how to prove it.

Question 42: What about infinitely many?

We do get counter examples.See Example.

Example 19.7.2

The Dirichlet function: $f(x) = \begin{cases} 1 \in \mathbb{Q} \\ 0, x \in \mathbb{I} \end{cases}$ For $f \in [0, 1]$ the claim is that f is not integrable on $[0, 1]$. Let $P = \{0 = x_0 < x_1 < \ldots < x_n = 1\}$ be an arbitrary partition of $[0, 1]$. $L(f;P) = 0$ for any subinterval by density of $\mathbb{I}$ $\exists r_k \in [x_{k-1}, x_k]$ with r_k irrational. That forces $m_k = 0 \forall k = 1, \ldots, n$ symmetric argument using density of $\mathbb{Q}$ that $M_k = 1$. This implies that $L(f;P) = 0$ and $U(f;P) = 1$, regardless of partition. The ϵ criterion fails.

Note:-

Bonus on the hw, example of $f : [a, b] \longrightarrow \mathbb{R}$ integrable but with ∞ many discontinuities.

Theorem 19.7.2

$f : [a, b] \longrightarrow \mathbb{R}$ bounded and if f is integrable on $[c, b]$ for every point $c \in (a, b)$ then f is integrable on $[a, b]$.

19.7.2 Properties of the integral

Theorem 19.7.3 2

Let $f : [a, b] \longrightarrow \mathbb{R}$ and $c \in (a, b)$ then f is integrable on $[a, b] \leftrightarrow f$ is integrable on $[a, c]$ and $[c, b]$. In this case $\int_a^b f = \int_a^c f + \int_c^b f$

Theorem 19.7.4 3 Properties of integral

Let $f, g : [a,b] \to \mathbb{R}$ be integrable functions. Then:

1. $f+g$ is integrable and $\int_a^b = \int_a^b f + \int_a^b g$
2. kf is integrable for k a constant, and $\int_a^b kf = k\int_a^b f$
3. Suppose $m \leqslant f(x) \leqslant M$ for all $x \in [a,b]$, then $m(b-a) \leqslant \int_a^b f \leqslant M(b-a)$
4. $f(x) \leqslant g(x), \forall x \in [a,b] \to \int_a^b f \leqslant \int_a^b g$
5. $|f|$ is integrable on $[a,b]$ and $|\int_a^b f| \leqslant \int_a^b |f|$

Proof: $(i), (ii)$ follow from HW11: f is integrable on $[a,b]$ if and only if $\exists$ sequence of partitions p_n on the closed $[a,b]$ such that $\lim U(f;P_n) - L(f;P_n) = 0$. In this case the integral

$\int_a^b f = \lim_{n\to\infty} U(f;P_n) = \lim_{n\to\infty} L(f;P_n)$.

(iii) is a fun one. For simplicity say that $m = \inf\{f(x) : x \in [a,b)\}$ and that $M = \sup\{f(x) : x \in [a,b]\}$. Recall that $L(f;P) \leqslant \int_a^b f \leqslant U(f;P)$ for all partitions P of $[a,b]$. Take (a,b), the smallest possible partition, this gives us $L(f;P) = m(b-a)$ and $U = (f;P) = M(b-a)$

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19.8 Thursday December 05

19.9 Reminders

1. $f : [a,b] \to \mathbb{R}$ bounded, $c \in (a,b)$ then f is integrable on $[a,b] \leftrightarrow f$ integrable on $[a,c]$ and $[c,b]$ and then $\int_a^b f = \int_a^c f + \int_c^b f$

Theorem 19.9.1

$f, g : [a,b] \to \mathbb{R}$ integrable.

1. $kf + lg$ is integrable for constant $k, l \in \mathbb{R}$ then $\int_a^b (kf+lg) = k\int_a^b f + l\int_a^b g$
2. If $m \leqslant f(x) \leqslant M_1 \forall x \in [a,b] \to m(b-a) \leqslant \int_a^b f \leqslant M(b-a)$
3. $|f|$ is integrable on $[a,b]$ and $|\int_a^b f| \leqslant \int_a^b |f|$

Note:-

If $b < a$ define $\int_a^b f = -\int_b^a f$

Definition 19.9.1: Zero Area Integral

$\int_a^a f = 0$.

Example 19.9.1

$f : \mathbb{R} \to \mathbb{R}$ integrable on every closed interval then $\int_a^b f = \int_a^c f + \int_c^b f, \forall a, b \in \mathbb{R}$.

Theorem 19.9.2 Fundamental Theorem of Calculus I

Let $f : [a, b] \to \mathbb{R}$ integrable. Suppose there exists $F : [a, b] \to \mathbb{R}$ such that F is continuous on $[a, b]$ differentiable on the open (a, b) and $F'(x) = f(x), \forall x \in (a, b)$. Then $\int_a^b f = F(b) - F(a)$.

Proof: Recall that $\int_a^b f = U(f) = L(f)$, for $U(f) = \inf\{U(f; P) : P \text{ a partition}\}$, and $L(f) = \sup\{L(f; P) : P \text{ a partition}\}$. We will show that if P is an arbitrary partition of $[a, b]$, then $L(f; P) \leqslant F(b) - F(a) \leqslant U(f; P)$.

This is a good enough strategy because, if $F(b) - F(a) \geqslant L(f; P)\forall P$, then $F(b) - F(a)$ is an upper bound of $\{L(f; P) : P\}$. This implies that $F(b) - F(a) \geqslant L(f)$. Similarly $F(b) - F(a) \leqslant U(f; P), \forall P \to F(b) - F(a)$ is a lower bound of $\{U(f; P) : P\}$. This implies that $F(b) - F(a) \leqslant U(f)$. Because $U(f) = L(f) \to F(b) - F(a) = U(f), L(f)$.

Let P be an arbitrary partition. $P = \{a = x_0 < x_1 < ... < x_n = b\}$, be a partition of $[a, b]$. The assumptions should make you think of MVT. F is continuous on $[a, b]$ and differentiable on the open $(a, b) \to F$ is continuous on $[x_{k-1}, x_k]$ and differentiable on each one of the open sub intervals for $k = 1, ..., n$. Apply MVT for F on $[x_{k-1}, x_k] \to \exists C_k \in (x_{k-1}, x_k)$ such that $F'(C_k) = \frac{F(x_k) - F(x_k)}{x_k - x_{k-1}}$. By our assumption $F'(C_k) = f(C_k)$. We will now cross multiply:

$f(C_k)(x_k - x_{k-1}) = F(x_k) - F(x_{k-1})$ for $k = 1, ..., n$. $m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$ and $M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$. Then we get the following:

$m_k \leqslant f(c_k) \leqslant M_k$ because $c_k \in [x_{k-1}, x_k] \to m_k(x_k - x_{k-1}) \leqslant f(c_k)(x_k - x_{k-1}) \leqslant M_k(x_k - x_{k-1})$ true for $k = 1, ..., n$.

Adding the inequalities gives us $\Sigma_{k=1}^n m_k(x_k - x_{k-1}) \leqslant \Sigma_{k=1}^n f(c_k)(x_k - x_{k-1}) \leqslant \Sigma_{k=1}^n M_k(x_k - x_{k-1})$. Immediately, we can now see that the middle is $\Sigma_{k=1}^n (F(x_k) - F(X_{k-1}) = F(b) - F(a)$ because the series is telescoping.

Then $L(f; P) \leqslant F(b) - F(a) \leqslant U(f; P)$.

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Theorem 19.9.3 Fundamental Theorem of Calculus II

Let $g : [a, b] \to \mathbb{R}$ integrable. Consider the function $G : [a, b] \to \mathbb{R}$ for $G(x) = \int_a^x g(t)dt$. Then

1. G is continuous on $[a, b]$ even if g is not.
2. If g is continuous, at some point $c \in [a, b]$, then G is differentiable at c and the derivative $G'(c) = g(c)$.

   x should be thought of as an endpoint that moves along the axis containing a, b.

Proof of 2 is omitted.

Proof: We will show that $\exists M > 0$ such that $|G(x) - G(y)| \leqslant M|x - y|\forall x, y \in [a, b]$.

g is integrable on $[a, b] \to |g| \to$ and $|\int_a^b g| \leqslant \int_a^b |g|$. and $|g|$ is bounded.

Let $M > 0$ be such that $|g(x)| \leqslant M \forall x \in [a, b]$. Let $x, y \in [a, b]$. Without loss of generality assume that $x \geqslant y$. Then $|\int_y^x g| \leqslant \int_y^x |g|$. We applied the same result from the beginning of today, except on the closed interval $[y, x]$.

This implies that $|\int_y^x g| \leqslant M$. Now we are done becuase we need to show that $|G(x) - G(y)| = |\int_a^x g(t)dt - \int_a^y g(t)dt| = |\int_a^x g + \int_y^a g| = |\int_y^x g|$. This results again from the the first theorem of today. This shows that $|G(x) - G(y)| \leqslant M(x - y) \to \epsilon\delta$ proof (Similar to Midterm 2 Question 6) that G is continuous.

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Example 19.9.2 (Application: Construction of Logarithm Function)

Consider the following function. $L : (0, \infty) \to \mathbb{R}$ with $L(x) = \int_1^x \frac{1}{t} dt$. We know from calculus that this should be $\log x$.

1. $L(1) = 0 = \int_1^1 \frac{1}{t} dt$

2. Because $f(t) = \frac{1}{t}$ is differentiable, and continuous on $(0, \infty)$. The Fundamental Theorem of Calculus Part 2 gives us that $L(x)$ is differentiable on $(0, \infty)$, and $L'(x) = \frac{1}{x}, \forall x \in (0, \infty)$.

Theorem 19.9.4 Property of Logarithm

For all $x, y > 0$ we have $L(x, y) = L(x) + L(y)$.

Proof: Let $y > 0$ be fixed. Consider the function $G(x) = L(xy) - L(x)$. Where $x \in (0, \infty)$. $G'(x) = 0$. By the chain rule and the sum rule, we get that $G'(x) = yL'(xy) - L'(x) = \frac{y}{xy} - \frac{1}{x} = 0$.

This gives us $G : (0, \infty) \to \mathbb{R}$ differentiable with $G'(x) = 0, \forall x > 0 \to G(x) = k = \text{constant}$. We make this arguemnent becuase $G : (0, \infty)$ is over an interval. $G(x)$ is constant so it is equal to $G(1)$. From earlier we know that $L(1) = 0$. Hence $G(1) = G(x) = L(y) - L(1) = L(y)$. This implies that $L(xy) - L(x) = L(y)$ as desired.

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Theorem 19.9.5 Property 2 of Logarithm

Range of log is $\mathbb{R}$. Done in Review.

Question 43: Why is the Logarithm function Invertible?

Solution: Claim that $\log(x)$ is an invertible function. We argue this because (switch to L notation). $L'(x) = \frac{1}{x} > 0, \forall x \in (0, \infty) \to L$ is strictly increasing on $(0, \infty)$. This implies that L is $1 : 1 \to L^{-1}$.

Definition 19.9.2: Exponential Function

We define $E : \mathbb{R} \to (0, \infty)$, $E = L^{-1}$ to be the exponential funciton.

Theorem 19.9.6

E is differentiable on every $y \in \mathbb{R}$ and $E'(y) = E(y)$.

Proof: $y = L(x)$. By a HW10 problem, we know that $E'(y) = \frac{1}{L'(x)}$ where $y = L(x)$.

$$E'(y) = \frac{1}{L'(x)} = \frac{1}{\frac{1}{x}} = x = L^{-1}(y) = E(y).$$

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Theorem 19.9.7

For all $x, y \in \mathbb{R}$ we have $E(x + y) = E(x)E(y)$.

Proof: Imitate the similar property of log. In this case we have to use quotient rule instead. Fix $y \in \mathbb{R}$ and consider the function, $F(x) = \frac{E(x+y)}{E(y)}$, allowed to divide because my function is everywhere positive never equal to zero.

Computing the derivative using quotient rule, chain rule we find $F'(x) = 0$.

$$\frac{E(x+y)}{E(x)} = \frac{E(x+y)E'(x) - E(x)E'(x+y)}{E(x)^2} = \frac{0}{E(x)^2}.$$

We evaluate at 0 $L(1) = 0 \to E(0) = 1 \to F(x) = F(0) \to E(x + y) = E(x)E(y)$.

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Note:-

We call the function exponential. We will declare $E(1) = e$. We check that $E(2) = E(1)E(1) = e^2$. Because of this $E(n) = e^n, \forall n \in \mathbb{N}$. We can extend this to all of the integers: $E(n) = e^n, \forall n \in \mathbb{Z}$. We extend this by writing $1 = E(0) = E(1 + (-1), E(-1) = e^{-1}$. Similarly $E(\frac{m}{n}) = e^{\frac{m}{n}}$